Introduction

In the seminar will expose results due to Carleman concerning asymptotic behaviour of eigenfunctions which arises while one reards the Laplace operator on the space $L^2(\Omega)$ where Ω is a bounded Dirichlet domain in $\mathbf{R}^2$. As a background a section is inserted about the planar Dirihclet problem where a result due to Boulignard and Perron is presented which shows that the Dirichlet problem has solutions for a quite extensive family of bounded open. A fundamnetal fact is that if Ω is a bounded Dirichlet domain , i.e. every $\phi \in C^0(\partial\Omega)$ has a harmonic extension to Ω, then there exists an orthonormal family $\{\phi_n\}$ of real-valued functions in the Hilbert space $L^2(\Omega)$ where

(*) $$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0$$

hold for a non-decreasiung sequence of positive real ,numbers $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Moreover, each ϕ_n is a continuous function which is identically zero in $\partial\Omega$. The family $\{\phi_n\}$ can be used to analyze various differential equations. An example comes from probability theory when a Browninan motion starts at time $t = 0$ at given a given point $z \in \Omega$. Then we can consider the function $t \mapsto \pi(z,t)$ which for every positive time $t > 0$ is the probablity that a Browninan path stays in Ω up to time t. Now π is a function of both t and z and via a trivial Taylor expansion one verifies that this function satisfies the PDE-equation

$$\frac{\partial\pi(z,t)}{\partial t} = \frac{1}{2} \cdot \Delta(\pi(z,t))$$

where the right hand side means that we apply the Laplace operator to the function $z \mapsto \pi(z,t)$ when the real (x,y) space is identified with the complex z-plane. When z is close to $\partial\Omega$ the probability to stay in Ω gets small and hence π satisfies the boundary condition

$$\lim_{z \to \partial\Omega} \pi(z,t) = 0 \quad : t > 0$$

Using one finds that this boundary value problem has the solution

$$\pi(z,t) = \sum c_n \cdot e^{-\lambda_n \cdot t/2} \cdot \phi_n(z)$$

Here $\{c_n\}$ are constants such that

$$\sum c_n \cdot \phi_n(z) = 1 \quad : z \in \Omega$$

expressing the fact that the Browninan motion stays inside Ω with high probability during small initial time intervals. Since $\{\phi_n\}$ is an orthonormal family in $L^2(\Omega$ one has

$$c_n = \iint_\Omega \phi_n(x,y)dxdy$$

The equations above illustrate that the eigenfunctions $\{\phi_n\}$ give insight into the pribelity which measure the life-time for a Brownian motion. Consider for example the mean-value for the time of survival when the motion starts at a point $z \in \Omega$. From the above the mean value is expressed by

$$\sum_{n=1}^{\infty} \int_0^\infty t \cdot \lambda_n/2 \cdot e^{-\lambda_n t/2}\, dt \cdot c_n \cdot \phi_n(z)$$

A computation shows that the sum above is equal to

(*) $$\sum c_n \cdot \phi_n(z)$$

Here the constants $\{c_n\}$ are independet of z so the point evaluations of the ϕ-functions determine the requiested mean value.

The example above illustrates why it is of interest to analyze asympotic behaviour of the sequence $\{\phi_n(p)\}$ for each fixed point p. The results in Section A give certain averaged asymptotic formulas for these point-evaluatioins. Before we enter the general material in Section A we insert some special observations related to the Laplace operator. To begin with we consider the 1-dimensional case where δ is reduced to the second order differential operator $\frac{s^2}{dx^2}$. Consider for example the interval $[0,\pi]$ and now we seek eigenfunctions $y(x)$ for which $y(0)=y(\pi)=0$. If n is a postiive integer we find that the sine-functiuon $y_n(x)=\sin(x)$ satisfies the equation

$$y_n''(x)+n^2y_n(x)=0$$

and via expansions of L^2-functions into Fourier series we conclude that $\{y_n(x)\}$ is is an orthogonal family in $L^2[0,\pi]$. Notice that

$$\int_0^\pi \sin^2(nx)\,dx=\frac{\pi}{2}$$

hold for every positive integer n and hence the functions

$$\phi_n(x)=\frac{\sqrt{2}}{\sqrt{\pi}}\cdot\sin(nx)$$

is an orthonormal family.

A limit formula. Consider some fixed number $0<a\pi$ and for each $N\geq 1$ we put

$$S_N(a)=\frac{1}{N}\cdot\sum_{n=1}^{n=N}\phi_n^2(a)$$

With these notations one has

$$(*)\qquad \lim_{N\to\infty} S_N(a)=\frac{1}{2\pi}$$

Thus, these averaged sums have a common limit for every $0<a<\pi$. The proof follows via the formula for trigonmetric functions expressed by

$$\sin^2(nx)=\frac{1}{2}\cdot(1-\cos(2nx))$$

and after one employs the fact that the series

$$\sum_{n=0}^{\infty} e^{-2ina}$$

converges for each $0<a<\pi$ where the sum is equal to $\frac{1}{1-e^{2ia}}$.

Passing to the 2-dimensional case we can consider the domain

$$\square=\{(x,y):0<x,y<\pi\}$$

Here we find eigenfunctions for the Laplace operator where

$$\Delta(\sin\,nx\cdot\sin\,my)+(n^2+m^2)\sin\,nx\cdot\sin\,y=0$$

and exactly as inte 1-dimensional case we get an orthonormal family in $L^2(\square)$ by the doubly-indexed functions

$$\phi_{n,m}(x,y)=\frac{2}{\pi}\cdot\sin\,nx\cdot\sin\,my$$

At this stage the reader is invited to check the contents of Theorem A.2 for the special Dirichlet domain $\square$ where the set of positive eigenvalues for Δ consists of positive integers of the form

$$n^2+m^2:\;n,m\geq 1$$

Limit formulas for more general PDE-operatos. On the real interval $[0,\pi]$ we can regard a second order differential operator

$$P(u)(x)=u''(x)+a(x)\cdot u(x)$$

where $a(x)$ is real-valud and of class C^1, i.e. continuosly differentiable and in addition periodic, i.e. $a(0)=a(\pi)$ and similarly for the first order derivative. Here P fails to be symmrric, i.e. its adjoint P^* becomes

$$P^*(u)(x)=u''(x)-a(x)f'(x)-a'(x)f(/x)$$

When $P \neq P^*$ the eigenvalues λ for which there exists a function $u(x)$ where $u(0) = u(\pi)$ and

$$u''(x) + a(x)u(x) - \lambda \cdot u(x) = 0$$

in general are non-real complex numbers. So now one is confronted with a delicate situation where conclusive limit formulas are hard - or even almost imposseible - to attain. However, it turns out that a general asymptotic formula hold for the real parts of the eigenvalues. More precisely, if the eigenvalues are arranged in such a way that thier real parts form a non-decreasisng sequence then

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \cdot \sum_{n=1}^{n=N} \mathfrak{Re}(\lambda_n) = 1$$

just as in the case when the a-function is identically zero.

Asymptotic formulas for eigenfunctions of the Laplace operator in Dirichlet domains

The subsequent results were presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934: Consider a bounded Dirichlet domain Ω $\mathbf{R}^2$, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. For every fixed point $p \in \Omega$ one regards the continuous function

$$q \mapsto \log \frac{1}{|p-q|} \quad : q \in \partial\Omega$$

which gives a unique harmonic function $x \mapsto H(p,x)$ in Ω such that

$$H(p,q) + \frac{1}{|p-q|} = 0 \quad : q \in \partial\Omega$$

A wellknown fact established by G. Neumann and H. Poincaré shows that the H-function is symmetric, i.e.

$$H(p,q) = H(q,p)$$

hold for each pair of points p, q in Ω. Greens' function is defined by

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

Notice that for each fixed $p \in \Omega$ it follows that the function $q \mapsto G(p,q)$ is super-harmonic and zero when $q \in \partial\Omega$. The minimum principle for superharmonic functions entails that

(i) $$G(p,q) > 0 \quad : p, q \in \Omega$$

and the reader shold check that

(ii) $$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence the linear operator $\mathcal{G}$ on the Hilbert space $L^2(\Omega)$ defined by

$$\mathcal{G}(\phi)(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q)\phi(q)dq$$

is of the Hilbert-Schmidt type and therefore compact on the Hilbert space $L^2(\Omega)$. Since the kernel positive the eigenvalues are positive, and wellknown facts about such nice integral operators give a sequence of pairwise orthogonal functions $\{\phi_n\}$ whose L^2-norms are one and

(1) $$\mathcal{G}(\phi_n) = \mu_n \cdot \phi_n$$

where $\{\mu_n\}$ is a non-increasing sequence of positive eigenvalues which tend to zero. Moreover, since the kernel $G(p,q)$ is positive it follows - again by general facts - that $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$, i.e. each real-valued function $f \in L^2(\Omega)$ has an expansion

(2) $$f = \sum a_n \cdot \phi_n \quad : a_n = \int_\Omega f_n(p) \cdot \phi_n(p)\, dp$$

Exercise. Verify that each ϕ-function extends to a continuous function on $\overline{\Omega}$ whose boundary values are zero.

Next, let Δ be the Laplace operator. Since $\frac{1}{2\pi} \cdot \log |z|$ is a fundamental solution it follows that

(3) $$\Delta \circ \mathcal{G}(f) = -f \quad : f \in L^2(\Omega)$$

Thus, the composed operator $\Delta \circ \mathcal{G} = -E$ where E is the identity operator. Put

$$\lambda_n = \mu_n^{-1}$$

Then (1) and (3) give

(4) $$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where we now have $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$.

With these notations we can announce Carleman's theorem.

A.1. Theorem. *For every Dirichlet domain Ω and each $p \in \Omega$ one has the limit formula*

$$(*) \qquad \lim_{N\to\infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

To prove this we consider some point $p \in \Omega$. Since every $\phi_n$is harmonic and has L^2-norm one, the reader can check that with a fixed p there exists a constant $C(p)$ such that

$$\phi_n(p)^2 \leq C(p) \quad : n = 1, 2, \ldots$$

Hence there exists the Dirichlet series

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

which is analytic in the half-space $\mathfrak{Re}\, s > 1$. We are going to prove the following result:

A. 2 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Remark. Theorem A.2 gives Theorem A.1 by a result due to Norbert Wiener in his article *Tauberian theorem* [Annals of Math.1932] which asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers tending to infinity and $\{a_n\}$ a sequence of non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem A. 2

Since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\text{(i)} \qquad \sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. For each λ outside the discrete set $\{\lambda_n\}$ we put

$$\text{(ii)} \qquad G(p, q; \lambda) = G(p, q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p) \cdot \phi_n(q)}{\lambda_n(\lambda - \lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\text{(iii)} \qquad \mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p, q; \lambda) \cdot f(q)\, dq$$

Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p, q, \lambda) = G(p, q; \lambda) - G(p, q)$$

Keeping p fixed we see that (ii) gives a function defined by

$$\text{(B.1)} \qquad F(p, \lambda) = \lim_{q\to p} F(p, q, \lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)}$$

From (i) and (B.1) it follows that this yields a meromorphic is a function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Use residue calculus to show the equality below in the half-space $\mathfrak{Re}\, s > 2$:

$$\Phi_p(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a-i\infty}^{a+i\infty} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi_p(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the right hand side is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To show this we are going to express $F(p, -x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2 - 1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$I_0(z) = \sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \tag{i}$$

$$I_1(z) = \sum_{m=0}^{\infty} \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma \tag{ii}$$

and γ is the usual Euler constant.

Next, with p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa)=K(\sqrt{\kappa}\cdot|p-q|)-H(p,q;\kappa)\quad:\kappa>0$$

From (F.1) the reader can verify the limit formula:

$$\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|)+\log\,|p-q|]=-\frac{1}{2}\cdot\log\kappa+\log 2-\gamma \tag{G.1}$$

where γ is Euler's constant. Next, the construction of $G(p,q)$ gives

$$F(p,-\kappa)=\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|)+\log\,|p-q|]+\lim_{q\to p}\,[H_p(q)+H(p,q,\kappa)] \tag{G.2}$$

The last term above has the "nice limit" $u_p(p)+H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|)+\log\,|p-q|]-\frac{1}{2}\cdot\log\kappa+\log 2-\gamma \tag{G.3}$$

where γ is Euler's constant.

H. Final part of the proof. Set

$$A=+\log 2-\gamma+H_p(p)$$

Then (G.1-3)) give

$$F(p,-\kappa)=-\frac{1}{2}\cdot\log\kappa+A+H(p,p;\kappa) \tag{H.1}$$

Above $\kappa>0$ and using $x=-\kappa$ in (E.2) we can proceed as follows. To begin with it is clear that

$$s\mapsto A\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty x^{-s}\,dx$$

is an entire function of s. Next, consider the function

$$\rho(s)=-\frac{1}{2}\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty\log x\cdot x^{-s}\,dx$$

Notice that the complex derivative

$$\frac{d}{ds}\int_a^\infty x^{-s}\,dx=-\int_a^\infty\log x\cdot x^{-s}\,dx$$

H.1 Exercise. Use the last equality to show that

$$\rho(s)-\frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem D.2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s\mapsto\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty H(p,p,\kappa)\cdot\kappa^{-s}\,d\kappa$$

Proof. When $\kappa>0$ the equation (F.1) shows that $q\mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

$$0\leq H(p,q;\kappa)\leq\max_{q\in\partial\Omega}\,K(\kappa|p-q|) \tag{i}$$

Next, since $p\in\Omega$ is fixed there is a positive number $\delta>0$ such that

$$|p-q|\geq\delta:q\in\partial\Omega$$

and it follows from the above that we can take $q=p$ and obtain

$$H(p,p;\kappa)\leq\int_1^\infty\frac{e^{-\delta\cdot t}}{\sqrt{t^2-1}}\,dt \tag{i}$$

If we choose some $0 < \alpha < \delta$ the reader can check that (i) yelds a constant b so that

$$H(p, p; \kappa) \leq B \cdot e^{-\alpha \cdot \kappa}$$

and finally it is clear that this exponential decay gives Lemma H.2.

The p^*-function.

We shall construct a special harmonic function which is used to get solutions of the Dirichlet problem. Let Ω be an open and connected set in $\mathbf{C}$ and consider the connected components of its closed complement. Let E be such a connected component which is not reduced to a single point. Let us then consider some closed and simple Jordan curve γ contained in Ω. For each point $a \in E$ there exists the winding number $\mathfrak{w}_a(\gamma)$ defined by the integer

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

If b is another point in the connected set E the continuity of the integer-valued winding number implies that

$$(*) \qquad \mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$$

This equality yields a single-valued analytic function in Ω given by the difference

$$\log(z-a) - \log(z-b)$$

Taking the exponential we find an analytic function $\psi(z)$ in Ω such that

$$e^{\Psi(z)} = \frac{z-a}{z-b}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$ and obtain the harmonic function in Ω defined by

$$(*) \qquad p(z) = \mathfrak{Re}\big(\frac{1}{\Psi(z)}\big) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2}$$

Next, we have

$$\mathfrak{Re}(\Psi(z)) = \operatorname{Log}|z-a| - \operatorname{Log}|z-b|$$

and since $\log|z-a| \to -\infty$ as $z \to a$ we see from (*) that

$$(**) \qquad \lim_{z\to a} p(z) = 0$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

$$(ii) \qquad \operatorname{Log}|z-a| - \operatorname{Log}|z-b| < -1 \quad : \quad |z-a| \leq \delta$$

From the above we can conclude:

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z\to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$0 < r < \delta \implies \max_{\{|z-a|=r\}\cap\Omega} p^*(z) < 0 \quad :$$

1. The Dirichlet Problem.

Introduction. Let Ω be a bounded open set in $\mathbf{C}$. No connectivity assumptions are imposed, i.e. neither Ω or $\partial\Omega$ have to be connected. To each $\phi \in C^0(\partial\Omega)$ we shall construct a harmonic function H_ϕ in Ω by a contruction due to Perron. Denote by $\mathcal{P}(\phi)$ the family of subharmonic functions $u(z)$ in Ω satisfying

$$\limsup_{z \to w} u(z) \leq \phi(w) \quad : \ w \in \partial\Omega \tag{0.1}$$

One verifies that the function defined for $z \in \Omega$ by

$$H_\phi(z) = \max_{u \in \mathcal{P}(\phi)} u(z) \tag{*}$$

is harmonic in Ω. A boundary point a is called Dirichlet regular if

$$\lim_{z \to a} H_\phi(z) = \phi(a) \quad : \ \phi \in C^0(\partial\Omega) \tag{**}$$

If (**) holds for every boundary point then Perron's solution extends to a continuous function on the closure $\bar{\Omega}$ and solves the Dirichlet problem with the prescribed boundary function ϕ. We shall prove that (**) holds under a certain geometric condition.

1.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of a in the closed complement $\mathbf{C} \setminus \Omega$ is not reduced to the singleton set $\{a\}$. Then (**) holds for every ϕ.*

The proof involves several steps. First we consider Perron's solution in the special case when

$$\phi(z) = |z - a|$$

which we denote by $H_a(z)$. Notice that the function

$$z \mapsto |z - a|$$

is subharmonic and hence Perron's function $H_a(z)$ satisfies

$$|z - a| \leq H_a(z) \quad : \quad z \in \Omega \tag{1}$$

Now we shall prove:

1.2 Boulignad's Lemma. *Let $a \in \partial\Omega$ satisfy the condition in Theorem 1.1. Then*

$$\lim_{z \to a} H_a(z) = 0$$

Proof. The assumption upon a and Theorem 0.1 gives a harmonic function $p^*(z)$ in Ω such that

$$\lim_{z \to a} p^*(z) = 0 \quad \text{and} \quad p^*(z) < 0 \quad : z \in \Omega \tag{1}$$

Next, let $\epsilon > 0$. Since $a \in \partial\Omega$ we find $0 < r \leq \epsilon$ such that the circle $|z - a| = r$ has a non-empty intersection Γ with Ω. Put

$$M = \max_{z \in \Omega} |z - a| \tag{2}$$

We can choose a compact subset Γ_* of Γ such that

$$\ell = \text{arc-length } (\Gamma \setminus \Gamma_*) \leq \frac{\epsilon}{M} \tag{3}$$

In the disc $D = \{|z - a| < r\}$ we find the harmonic function $V(z)$ whose boundary values on $|z - a| = r$ are zero outside the open set $\Gamma \setminus \Gamma_*$ while $V = M$ holds on $\Gamma \setminus \Gamma_*$. Next, since Γ_* is a compact subset of Ω and $p_a^* < 0$ in Ω there exists $\delta > 0$ such that

$$p^*(z) \leq -\delta \quad : \quad z \in \Gamma_* \tag{4}$$

Set

$$B(z) = V(z) - \frac{M}{\delta} \cdot p^*(z) \tag{5}$$

This is a harmonic function in $\Omega \cap D$ and the construction of V together with (4) give

$$B(z) \geq M \quad : \quad z \in \Gamma \tag{6}$$

Next, in the the open set $U = \Omega \cap D$ we have the subharmonic function

(7) $$g = H_a - B$$

Since $|z-a| \leq \epsilon$ holds in the closed disc in $\bar{D}$ we have

(8) $$\limsup_{z \to w} H_a(z) \leq \epsilon \quad : w \in \bar{D} \cap \partial\Omega$$

and (2) implies that $H_a(z) \leq M$ holds in Ω which gives

(9) $$\limsup_{z \to w} H_a(z) \leq M \quad : w \in \Gamma$$

At this stage we use the set-theoretic inclusion

(10) $$\partial(D \cap \Omega) \subset \Gamma \cup (\bar{D} \cap \partial\Omega)$$

Hence (6) together with (7-8) entail that

(10) $$\limsup_{z \to w} H_a(z) - B(z) \leq \epsilon \quad : w \in \partial(\Omega \cap D)$$

The maximum principle applied to the subharmonic function $H - B$ in $\Omega \cap D$ and (10) give

(11) $$\limsup_{z \to a} H_a(z) \leq \epsilon + \limsup_{z \to a} B(z) = \epsilon + V(a) + \limsup_{z \to a} p^*(z) = \epsilon + V(a)$$

where (1) gives the last equality. Finally, the mean-value formula for the harmonic function V and (3) entail that

(12) $$V(a) = \ell \cdot M \leq \epsilon$$

Hence the limes superior in the left hand side of (11) is $\leq 2\epsilon$ and since $\epsilon > 0$ was arbitrary small we get $\limsup_{z \to a} H_a(z) \leq 0$ which finishes the proof of Boulignad's lemma.

§ 1.3 Proof of Theorem 1.1.

Let $\phi \in C^0(\partial\Omega)$ with the Perron solution $H_\phi(z)$. If c is a constant it is clear that $H_{\phi - c} = H_\phi - c$. Replacing ϕ by $\phi(z) - \phi(a)$ we may therefore assume that $\phi(a) = 0$ and it remains to show that

(1) $$\lim_{z \to a} H_\phi(z) = 0$$

First we consider the limes superior and show that

(2) $$\limsup_{z \to a} H_\phi(z) \leq 0$$

To get (2) we take some $\epsilon > 0$ and the continuity of ϕ gives $\delta > 0$ such that

$$\phi(z) \leq \epsilon \quad : \quad z \in \partial\Omega \cap D_a(\delta)$$

Put $M^* = \max_{z \in \partial\Omega} |\phi(z)|$ and define the harmonic function in Ω by

$$g^*(z) = \epsilon + \frac{M^*}{\delta} \cdot H_a(z)$$

Since $H_a(z) \geq |z - a|$ we have:

$$\liminf_{z \to b} g^*(z) \geq M^* \quad : \quad b \in \partial\Omega \setminus D_a(\delta)$$

At the same time $g^*(z) \geq \epsilon$ for every $z \in \Omega$ so $g^* \geq \phi$ on the whole boundary and the maximum principle for harmonic functions gives:

$$u \leq g^* \quad : u \in \mathcal{P}(\phi)$$

The construction of H_ϕ entails that $H_\phi \leq g^*$ holds in Ω which implies that

$$\limsup_{z \to a} H_\phi(z) \leq \limsup_{z \to a} g^*(z) = \epsilon$$

where the last equality follows from Boulignad's Lemma. Since ϵ can be arbitrary small we get (2). There remains to show that

(3) $$\liminf_{z \to a} H_\phi(z) \geq 0$$

To prove (3) we put

$$g_*(z) = -\epsilon - \frac{M^*}{\delta} \cdot H_a(z)$$

It is clear that

$$\limsup_{z \to \xi} g_*(z) \leq \phi(\xi) \quad \text{for all boundary points } \xi \in \partial\Omega$$

Hence $g_* \in \mathcal{P}(\phi)$ which gives $g_* \leq H_\phi$. So now we have

$$\liminf_{z \to a} H_\phi(z) \geq \liminf_{z \to a} g_*(z) = -\epsilon$$

Since ϵ can be arbitrary small we get (3) and Theorem 1.1 is proved.