Lecture about unbonded self-adjoint operators

Given a complex Hilbert space H which to every pair of vectos x, y yields the inner product $\langle x, y\rangle$ a densely defined linear operator A is symmetric if

$$\langle Ax, y\rangle = \langle x, Ay\rangle : \; x, y \in \mathcal{D}(A)$$

where $\mathcal{D}(A)$ is the domain of definition of A. Following a construction introdued in full generality by Hermann Weyl around 1910 one gets the adjoint operator A^* where a vector y belongs to $\mathcal{D}(A^*)$ if there exists a constant $C(y)$ such that

$$|\langle Ax, y\rangle| \leq C(y) \cdot ||x|| : \; x \in \mathcal{D}(A)$$

and then A^*y is the vector for which

$$\langle x, y\rangle = \langle x, A^*y\rangle : \; x \in \mathcal{D}(A)$$

The symmetry of A entails that A^* is an extension, i.e. one has an inclusion of graphs:

(*) $$\Gamma(A) \subset \Gamma(A^*)$$

One checks that $\Gamma(A^*)$ has a closed graph and hence A^* has a spectrum $\sigma(A^*)$ which is a closed subset of $\mathbf{C}$. In the special case when A is a bounded operator with $\mathcal{D}(A) = \mathcal{H}$ the symmetry implies that $A = A^*$ and $\sigma(A)$ is a compact subset of the real line. For an unbounded densely defined symmetric operator the situation is more complicated. In fact, around 1910 Hermann Weyl found "ugly examples" where A is a second order differential operator acting on a dense subspace of $L^2(\mathbf{R})$. while $\sigma(A^*)$ is the whole complex plane. This led Torsten Carleman to analyze more general cases. His monograph *Sur les équations singulières à noyau réel et symmetrtique* published in Uppsala 1923 contains a study of unbounded self-adjoint operators which culminates in the spectral decomposition which extends David Hilbert's classic result of spectral resolutions of bounded and self-adjoint operators. More precisely Carleman realised that A has a self-adjoint extension if and only if the inclusion (*) gives equality when we pass to the closure of $\Gamma(A)$, i.e. when

(**) $$\overline{\Gamma(A)} = \Gamma(A^*)$$

In my talk I explain how (**) yields the spectral theorem for unbounded self-adjoint operators which after is followed by some examples. One is the Schrödinger operator $L = \Delta + c(x, y, z)$ where Δ is the Laplace operator in $\mathbf{R^3}$ and $c(x, y, z)$ a real-valued and locally square integrable function. It is clear that L is densely defined on the complex Hilbert space $L^2(\mathbf{R^3})$ and restricted to the dense subspace $C_0^\infty(\mathbf{R^3}$ one has symmetry i.e.

$$\iiint Lf) \cdot \bar{g}\, dxdydz = \iiint f \cdot L(\bar{g})\, dxdydz$$

hold when f and g are test-functions. There exist "ugly cases" where L fails to have a self-adjoint extension while self-adjointness hold for certain c-functions. A *sufficient condition* for L to admit a self-adjoint extension is that the c function satisfies the limit condition:

(1) $$\text{Lim.sup}_{x^2+y^2+z^2 \to +\infty}\, c(x, y, z) < +\infty$$

This result is proved in Carleman's article *Sur la theorie mathématique de l'équation de Schrödinger* published in Arkiv för matematik 1934. I my lecture I will present the special case when the c-function is a Newtonian potential, i.e

$$c(p) = \sum_{\nu=1}^{\nu=N} \frac{m_\nu}{|p - p_\nu|}$$

where $p_1, \ldots, p_N$ is a finite set of mass-points in $\mathbf{R^3}$.