ANALYSIS ON MANIFOLDS
Global Calculus of Variations and Differential Equations

Lecturer: Professor Olga Rossi

7.5 hp.
Thursdays 15:15-17:00 (start 23/1), room 16 building 5.
This is a PhD level course, also suitable for interested Master students.

The course is an introduction to the calculus of variations and differential equations on manifolds. It covers Lagrangian systems in jet bundles, variational equations and the inverse problem of the calculus of variations, symmetries and conservation laws, Hamiltonian systems, integration methods based on symmetries. We shall be interested also in differential equations with differential constraints (geometric mechanics of non-holonomic systems).

Students can assist to design the course according to their interests in topics listed below.

Tentative Contents:

1. Reminder of analysis on manifolds:
Smooth manifolds and smooth mappings, vector fields and differential forms, immersions and embeddings, submersions.

2. First order differential equations on manifolds:
Vector fields, flows, one-parameter groups of transformations. Vector distributions of constant and non-constant rank, integral manifolds, Frobenius theorem, Sussmann-Viflyantsev theorem. Symmetries of distributions, first integrals. Closed two-forms, characteristic distribution, Darboux theorem. A few words on exterior differential systems on manifolds, Cauchy characteristics and formal integrability.

3. Jet bundles:
Fibred manifolds, jet prolongations of sections, the contact structure, canonical decomposition of differential forms, jet fields and connections. Differential equations in jet bundles, semispray connections, the Vessiot distribution, symmetries.

4. Lagrangian systems and variational equations:
Lepage forms, the first variation formula, extremals. Holonomic constraints in mechanics and geometry. Locally variational forms, the inverse problem of the calculus of variations, the relationship between the Euler-Lagrange operator and the exterior derivative, the Helmholtz form.

5. Hamiltonian systems (ODEs, mechanics):

6. Symmetries and conservation laws:
Noether and Noether-Bessel-Hagen equation, Noether theorem. Classification of symmetries, relations between different symmetries of variational structures. Symmetries of the Helmholtz form
and their meaning in the theory of differential equations.

7. Hamiltonian systems (PDEs, field theory):
Poincaré-Cartan form, De Donder-Hamilton equations, duality in field theory, Ehresmann connections for regular Hamiltonian systems, Lepage forms of higher degree and their related Hamiltonian systems. Applications in general relativity and to Lagrangians of electromagnetic type.

8. Geometric integration methods for regular and semiregular variational ODEs:
The Liouville theorem, Jacobi complete integrals and Hamilton-Jacobi integration method, canonical transformations, fields of extremals and the generalized Van Hove theorem, Hamilton-Jacobi distributions.

9. Mechanical systems with non-holonomic constraints:
The constraint structure, differential equations with non-holonomic constraints, the constraint variational principle, symmetries and conservation laws. Applications in geometric mechanics.

RECOMMENDED READING:
R. Narasimhan, Analysis on Real and Complex Manifolds, North-Holland, Amsterdam, 2006
selected articles

PREREQUISITES:
Topology, linear algebra, calculus, differential equations.
Recommended course: Differential geometry.

EXAMINATION:
Oral exam consisting of a short seminar talk on a topic covered by the course.