ABSTRACT

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A one phase Hele-Shaw flow, described by a domain $D_t = \{x : u(x) > 0\} \cup D_0$ for $t \geq 0$, is the flow of a liquid injected at a constant rate in the separation between two narrowly separated parallel planes. This thesis deals with the formulation and proof of existence for a multiple phase Hele-Shaw flow exhibiting separation of the phases. A smooth version of the problem, depending on the parameter $\varepsilon > 0$, has been considered giving rise to the equations in $H^{-1}(\Omega)$

$$-\Delta u^i + (1 - \chi_{D_0^i})\beta_\varepsilon(u^i) = t\mu^i - \frac{1}{\varepsilon}\sum_{j \neq i} B_\varepsilon(u^j)\beta_\varepsilon(u^i) \text{ for } i = 1, \ldots, m \tag{1}$$

where $D_0^i \subset \Omega \subset \mathbb{R}^n$, $\mu^i \in H^{-1}(\Omega)$, β_ε is a mollification of the Heaviside step function, $B(s) = \int_0^s \beta_\varepsilon(s')ds'$, $u = (u^i)_{i=1,\ldots,m}$ a vector with components $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is the dual of the Sobolev space $H_0^1(\Omega)$. We show that the smooth problem has a solution $[0, \infty) \ni t \mapsto u_{t;\varepsilon} \in H_0^1(\Omega; \mathbb{R}^m)$ using a variational technique. Upon letting $\varepsilon \to 0^+$, for fixed t, the solution $u_{t;\varepsilon}$ converges weakly to $u_t \in H_0^1(\Omega; \mathbb{R}^m)$ solving

$$-\Delta u_t^i + (1 - \chi_{D_0^i})\chi_{\{u_t^i > 0\}} = t\mu^i - \kappa_t^i \text{ in } H^{-1}(\Omega), \tag{2}$$

for some non-negative elements $\kappa_t^i \in H^{-1}(\Omega)$ having support on ∂D_t^i . Furthermore the phases represented by the components of u_t are separated in the sense that the overlap of any two distinct phases has vanishing n-dimensional Lebesgue measure i.e.

$$\left| \text{supp } u_t^i \cap \text{supp } u_t^j \right| = 0 \text{ for} i \neq j. \tag{3}$$

We also touch upon a formulation of the multiple phase Hele-Shaw flow which would, beyond separation of the phases, provide freezing of the intersecting boundary $\Gamma_t^{ij} = \partial D_t^i \cap \partial D_t^j$ of two phases. This formulation of the problem tries to incorporate memory in to the system via means of an integration over previous states.