# From the Fundamental Theorem of Algebra to Astrophysics: a "Harmonious" Path

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ABSTRACT

The Fundamental Theorem of Algebra first rigorously proved by Gauss states that each complex polynomial of degree $n$ has precisely $n$ complex roots. In recent years various extensions of this celebrated result have been considered. We shall discuss the extension of the FTA to harmonic polynomials of degree $n$. In particular, the 2003 theorem of D. Khavinson and G. Swiatek that shows that the harmonic polynomial $\bar{z} - p(z), deg\, p = n > 1$ has at most $3n - 2$ zeros as was conjectured in the early 90ś by T. Sheil-Small and A. Wilmshurst. More recently L. Geyer was able to show that the result is sharp for all $n$.

In 2006 G. Neumann and D. Khavinson showed that the maximal number of zeros of rational harmonic functions $\bar{z} - r(z), deg\, r = n > 1$ is $5n - 5$. It turned out that this result confirmed several consecutive conjectures made by astrophysicists S. Mao, A. Petters, H. Witt and, in its final form, the conjecture of S. H. Rhie that were dealing with the estimate of the maximal number of images of a star if the light from it is deflected by $n$ co-planar masses. The first non-trivial case of one mass was already investigated by A. Einstein around 1912.

We shall also discuss the problem of gravitational lensing of a point source of light, e.g., a star, by an elliptic galaxy, more precisely the problem of the maximal number of images that one can observe. Under some more or less "natural" assumptions on the mass distribution within the galaxy one can prove that the number of visible images can never be more than four in some cases and six in the other. Interestingly, the former situation can actually occur and has been observed by astronomers. Still there are much more open questions than there are answers.