Glimpses from work by Carleman

Contents

Introduction. The lecture is devoted to spectral properties of second order elliptic PDE-operators. To illustrate the methods we are content to treat a special case while elliptic operators with variable coefficients are treated in separate notes devoted to mathematics by Carleman. However, § E in the appendix contains a construction of fundamental solutions to elliptic second order operators in $\mathbf{R}^3$ based upon Carleman's lectures at Institute Mittag Leffler in 1935 which might be of interest to some readers even though it will not be covered during my lecture. As background the appendix contains a section which explains Gustav Neumann's fundamental construction from 1879 of resolvents to densely defined linear operators, and at the end of § A we also recall Hilbert's spectral theorem for bounded normal operators on Hilbert spaces.

Let us now announce a major result which will be exposed in the lecture. In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \iint G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity. The result below was presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934:

0. Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

Remark. The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorophic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. For the reader's convenience we insert details of the proof in § D of the appendix which illustrates the "spirit" of the lecture.

Self-adjoint extensions of $\Delta + c(p)$. Here we consider $\mathbf{R}^3$ with points $p = (x, y, z)$ and Δ is the Laplace operator, while $c(p)$ is a real-valued and locally square integrable function. The linear operator

$$u \mapsto L(u) = \Delta(u) + c \cdot u$$

is defined on test-functions and hence densely defined on the Hilbert space $L^2(\mathbf{R}^3)$. In the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923], Carleman established spectral resolutions for unbounded self-adjoint operators on a Hilbert space together with conditions that densely defined symmetric operators have self-adjoint extensions, The lecture will describe the major steps of a result due to Carleman which asserts that the operator L has a self-adjoint extension under the condition that

(*) $$\limsup_{p \to \infty} c(p) \leq M$$

hold for some constant M. A special case occurs when $c(p)$ is a Newtonian potential

(**) $$c(p) = \sum \frac{\alpha_\nu}{|p - q_\nu|} + \beta$$

where $\{q_\nu\}$ is a finite set of points in $\mathbf{R}^2$ while $\{\alpha_\nu\}$ and β are positive real numbers. So here one encounters the Bohr-Schrödinger equation which stems from quantum mechanics. With c as in (**) the requested self-adjoint extension is easily verified while the existence of a self-adjoint extension when (*) holds requires a rather involved proof.

An asymptotic expansion. Consider the Schrödinger equation

$$i \cdot \frac{\partial u}{\partial t} = \Delta(u) + c \cdot u$$

where we assume that $L^\Delta + c$ has a self-adjoint extension. One seeks solutions $u(x, t)$ defined when $t \geq 0$ and $x \in \mathbf{R}^3$ with an initial condition $u(x, 0) = f(x)$ for some $f \in L^2(\mathbf{R}^3)$. The solution is given via the spectral function associated with the L-operator. So the main issue is to get formulas for the spectral function of $\Delta + c$. In Carleman's cited lecture from 1934 an asymptotic expansion is given for this spectral function which merits further study since one nowadays can investigate approximative solutions numerically by computers.

§ A. Linear operators and spectral theory.

Let X be a Banach space and $T\colon X \to X$ a linear and densely defined operator whose domain of definition is denoted by $\mathcal{D}(T)$. In general T is unbounded:

$$\max_{x\in\mathcal{D}(T)} ||T(x)|| = +\infty$$

where the maximum is taken over unit vectors in $\mathcal{D}(T)$. The graph is defined by

$$\Gamma(T) = \{(x, Tx) : x \in \mathcal{D}(T)$$

If $\Gamma(T)$ is closed in $X \times X$ one says that T has a closed graph.

A.1 Invertible operators. A densely defined operator T has a bounded inverse if the range $T(\mathcal{D}(T))$ is equal to X and there exists a positive constant c such that

(i) $$||T(x)|| \geq c \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Since $T(\mathcal{D}(T)) = X$, (i) gives for each $x \in X$ a unique vector $R(x) \in \mathcal{D}(T)$ such that

(ii) $$T \circ R(x) = x$$

Moreover, the inequality (i) gives

(iii) $$||R(x)|| \leq c^{-1} \cdot ||x|| \quad : x \in X$$

and when R is applied to the left on both sides in (ii), it follows that

(iv) $$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

A.2 The spectrum $\sigma(T)$. Let E be the identity operator on X. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. If it fails to be invertible one says that λ is a spectral point of T and denote this set by $\sigma(T)$. If $\lambda \in \mathbf{C} \setminus \sigma(T)$ the inverse to $\lambda \cdot E - T$ is denoted by $R_T(\lambda)$ and called a Neumann resolvent to T. By the construction in (A.1) the range of every Neumann resolvent is equal to $\mathcal{D}(T)$ and one has the equation:

(A.2.1) $$T \circ R_T(\lambda)(x) = R_T(\lambda) \circ T(x) \quad : x \in \mathcal{D}(T)$$

A.3 Neumann's equation. *Assume that $\sigma(T)$ is not the whole complex plane. For each pair $\lambda \neq \mu$ outside $\sigma(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and*

(*) $$R_T(\mu)R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

(i) $$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (*) which at the same time this shows that the resolvent operators commute.

A.4 The position of $\sigma(T)$. Assume that $\mathbf{C} \setminus \sigma(T)$ is non-empty. We can write (*) in the form

(1) $$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu)$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E + (\lambda - \mu)R_T(\mu))$ is invertible which implies that

(A.4.1) $$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\}$$

Hence one recovers $\sigma(T)$ via the spectrum of any given resolvent operator. Notice that (A.4.1) holds even when the open component of $\sigma(T)$ has several connected components.

A.4.2 Example. Suppose that $\mu = i$ and that $\sigma(R_T(i))$ is contained in a circle $\{|\lambda + i/2| = 1/2\}$. If $\lambda \in \sigma(T)$ the inclusion (A.4.1) gives some $0 \leq \theta \leq 2\pi$ such that

$$\frac{1}{i-\lambda} = -i/2 + 1/2 \cdot e^{i\theta} \implies 1 - i \cdot e^{i\theta} = \lambda(e^{i\theta} - i) \implies$$

$$\lambda = \frac{2 \cdot \cos\theta}{|e^{i\theta} - i|^2} \in \mathbf{R}$$

A.4.3 Neumann series. Let λ_0 be outside $\sigma(T)$ and construct the operator valued series

$$S(\zeta) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1} \tag{1}$$

It is clear that (1) converges in the Banach space of bounded linear operators when

$$|\zeta| < \frac{1}{||R_T(\lambda_0)||} \tag{2}$$

Moreover, the series expansion (1) gives

$$(\lambda_0 + \zeta - T) \cdot S(\zeta) = (\lambda_0 - T) \cdot S(\zeta) + \zeta S(\zeta) = E \tag{3}$$

Hence $S(\zeta) = R_T(\lambda_0 + \zeta)$ and the locally defined series in (1) entail the complement of $\sigma(T)$ is open where $\lambda \mapsto R_T(\lambda)$ is an analytic operator-valued function. Finally (*) in (A.3) and a passage to the limit as $\mu \to \lambda$ shows that this analytic function satisfies the differential equation

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda) \tag{**}$$

B. Bounded normal operators on Hilbert spaces.

The result below stems from Hilbert's work on integral equations from 1904. Let $\mathcal{H}$ be a complex Hilbert space. The inner product of a pair of vectors x, y is denoted by (x, y) and we recall that

$$(y, x) = \overline{(x, y)}$$

If T is a bounded linear operator on $\mathcal{H}$ its adjoint T^* satisfies $(Tx, y) = (x, T^*y)$. A bounded linear operator R is normal if it commutes with its adjoint, i.e. if $RR^* = R^*R$.

B.1 The spectral measure of normal operators. Let R be bounded and normal. Now $\sigma(R)$ is a compact set in the complex plane and denote by $\mathfrak{M}$ the family of all Riesz measures supported by $\sigma(R)$.

Theorem. *There exists a map from $\mathcal{H} \times \mathcal{H}$ to $\mathfrak{M}$ which to each pair of vectors x, y assigns $\mu_{\{x,y\}}$ in $\mathfrak{M}$ such that the following hold for every pair of non-negative integers*

$$(R^m x, R^k y) = \int \lambda^m \cdot \bar{\lambda}^k \cdot d\mu_{\{x,y\}}(\lambda) \tag{*}$$

Remark. If ϕ is a bounded Borel function on $\sigma(R)$ it can be integrated in the sense of Stieltjes with respect to the Riesz measures $\{d\mu_{\{x,y\}}$ which yields a bounded linear operator $\widehat{\phi}$ such that

$$(\widehat{\phi}(x), y) = \int \phi(\lambda) \cdot d\mu_{\{x,y\}}(\lambda) \tag{**}$$

hold for every pair x, y in $\mathcal{H}$. Moreover, (**) in an isometry which means that the operator norm $\widehat{\phi}$ is the maximum norm of $|\phi|$ and the spectrum of the bounded operator $\widehat{\phi}$ is the closure of the range $\phi(\sigma(R))$. It means that $\phi \mapsto \widehat{\phi}$ is an isomorphism of the commutative Banach algebra of bounded Borel functions on $\sigma(R)$ and a commutative and closed subalgebra of $L(\mathcal{H})$. In particular every Borel set $e \subset \sigma(R)$ yields the operator $\widehat{\chi_e}$ which is an idempotent and has a spectrum contained in the compact closure of the Borel set e.

C. Symmetric operators.

A densely defined linear operator S on a Hilbert space is symmetric if

$$(Sx, y) = (x, Sy) : x, y \in \mathcal{D}(S)$$

Now there exists the subspace $\mathcal{D}^*$ of vectors y for which there exists a constant $C(y)$ such that

$$|(Sx, y)| \leq C(y) \cdot ||x|| : x \in \mathcal{D}(S)$$

Since Hilbert spaces are self-dual and $\mathcal{D}(S)$ is dense, each $y \in \mathcal{D}^*$ gives a unique vector S^*y such that

$$(Sx, y) = (x, S^*y) : x \in \mathcal{D}(S)$$

So S^* is a new linear operator where $\mathcal{D}(S^*) = \mathcal{D}^*$. The symmetry of S entails that

$$\Gamma(S) \subset \Gamma(S^*)$$

One shows easily that S^* has a closed graph and hence

(i) $$\overline{\Gamma(S)} \subset \Gamma(S^*)$$

Definition. *A densely defined and symmetric operator S is of type I if equality holds in (i).*

Exercise. Let S be symmetric and suppose in addition that it has a closed graph. Show that S is of type I if and only if the two eigenvalue equations

$$Sx = i \cdot x : Sx = -i \cdot x$$

have no non-zero solutions.

The Cayley transform. Let S be symmetric of Type 1. Using the exercise above one shows that there exists Neumann's resolvent

$$R = R_S(i) = (iE - S)^{-1}$$

The equality $S = S^*$ entails that R is a normal operator and we can apply Hilbert's spectral theorem to R. This gives a spectral measure of S-operator. More precisely, there exists map from $\mathcal{H} \times \mathcal{H}$ into the space of complex-valued Riesz measure on the real line. Here the total variations satisfy

$$||\mu_{x,y}|| = \int_{-\infty}^{\infty} |d\mu_{\{x,y\}}(s)| \leq ||x|| \cdot ||y||$$

Moreover, $\mu_{\{x,x\}}$ are non-negative measures for each $x \in \mathcal{H}$ and a vector x belongs to $\mathcal{D}(S)$ if and only if

$$\int_{-\infty}^{\infty} s^2 \cdot d\mu_{\{x,x\}}(s) < \infty$$

Finally one has the equations

$$(Sx, Sy) = \int_{-\infty}^{\infty} s^2 \cdot d\mu_{\{x,y\}}(s) \quad : \; x, y \in \mathcal{H}$$

Remark. In many applications S is a PDE-operator defined on a suitable family of square integrable functions in a domain Ω of $\mathbf{R}^n$ for some positive integer n. To determine if S is symmetric is an easy task since it suffices to consider its restriction to the dense subspace of test-functions in Ω. But to verify that a symmetric PDE-operator is of Type I can be quite involved, and when S if of Type I there remains to investigate the normal operator $R_S(i)$ above and proceed to determine the spectral measure of S expressed by the μ-map above. For a quite extensive family of elliptic operators the spectrum of S discrete and one is led to analyze its asymptotic behaviour. In such studies the symmetry condition can often be relaxed, i.e. it suffices that the leading part of the PDe-operator is symmetric,

Example. Let $n = 3$ and consider a PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω with a C^1-boundary. Here one has the symmetry $a_{pq} = a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2 \cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u which is a C^2.-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa \geq \kappa_0$ a Green's function $G(x, y; \kappa)$ such that

(i) $$(L - \kappa^2)(\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

(ii) $$f \mapsto -\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

is Neumann's resolvent to the densely defined operator $L - \kappa^2$ on the Hilbert space $L^2(\Omega)$. After a detailed study of these G-functions, Carleman established an asymptotic formula for the discrete sequence of eigenvalues $\{\lambda_n\}$. In general they are complex but arranged so that the absolute values increase. To begin with one proves rather easily that they are "almost real" in the sense that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)| \leq C \cdot (\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

(*) $$\lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and was extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals, and so on. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x, \xi; \kappa)$ to the PDE-operators $L - \kappa^2$ which is exposed in § E.

§ D. Proof of Theorem 0.

Let Ω be a bounded and Dirichlet regular domain. Let $p \in \Omega$ be kept fixed and consider the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \, : \, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$G(p,q) = \log \frac{1}{|p-q|} - u_p(q) \tag{1}$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

$$\mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq \tag{2}$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega \times \Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this one can deduce the following:

D.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$\Delta(\mathcal{G}_f) = -f \tag{*}$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are theorefore led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where its domain of definition $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By the construction of resolvent operators in § 1 this means that $-\mathcal{G}$ is Neumann's inverse of S.

Exercise. Show that S has a closed range and in addition it is self-adjoint, i.e. $S = S^*$.

The spectrum of S. A wellknown result asserts that there exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction. More preciesely there is a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ and

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{i}$$

Let us remark that (i) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n \tag{ii}$$

So above $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. Eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times. Now we announce a result about the values taken by the eigenfunctions.

Now we begin the proof of Theorem 0, i.e. we must show that the following hold for each point $p \in \Omega$:

$$(*) \qquad \lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

To prove (*) we consider the Dirichlet series for each fixed $p \in \Omega$:

$$(**) \qquad \Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

It is easily seen that $\Phi_p(s)$ is analytic in a half-space $\mathfrak{Res} > b$ for a large b. Less trivial is the following:

D.2 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem D.2 gives (*) via a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem D.2

Since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\text{(i)} \qquad \sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. For each λ outside $\{\lambda_n\}$ we set

$$\text{(ii)} \qquad G(p,q;\lambda) = G(p,q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\text{(iii)} \qquad \mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p,q;\lambda) \cdot f(q)\, dq$$

A. Exercise. Use that the eigenfunctions$\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p,q,\lambda) = G(p,q;\lambda) - G(p,q)$$

Keeping p fixed we see that (ii) gives

$$\text{(B.1)} \qquad \lim_{q\to p} F(p,q,\lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)}$$

Set

$$F(p,\lambda) = \lim_{q\to p} F(p,q,\lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a-i\infty}^{a+i\infty} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p,-x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p,-x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^{\infty} \frac{e^{-zt}}{\sqrt{t^2-1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$\text{(i)} \qquad I_0(z) = \sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \cdot z^{2m}$$

$$\text{(ii)} \qquad I_1(z) = \sum_{m=0}^{\infty} \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p,q;\kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p,q;\kappa) = K(\sqrt{\kappa}|p-q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa) = K(\sqrt{\kappa}\cdot|p-q|) - H(q;\kappa) \quad : \kappa > 0$$

Next, the construction of $G(p,q)$ gives

(G.1) $$F(p,-\kappa) = \lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] + \lim_{q\to p}\,[u_p(q) + H(p,q,\kappa)]$$

The last term above has the "nice limit" $u_p(p) + H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

(G.2) $$\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] = -\frac{1}{2}\cdot\log\kappa + \log 2 - \gamma$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p,-\kappa) = -\frac{1}{2}\cdot\log\kappa + A + H(p,p;-\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty x^{-s}\,dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2}\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty \log x\cdot x^{-s}\,dx$$

Notice that the complex derivative

$$\frac{d}{ds}\int_a^\infty x^{-s}\,dx = -\int_a^\infty \log x\cdot x^{-s}\,dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem D.2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin\,\pi s}{2\pi^2}\cdot\int_a^\infty H(p,p,\kappa)\cdot\kappa^{-s}\,d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q\in\partial\Omega}\,K(\kappa|p-q|)$$

With $p\in\Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q\in\partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

§ E. Fundamental solutions to second order Elliptic operators.

The conclusive result in Theorem 1.9 gives sharp estimate for the fundamental solutions. The subsequent constructions are based upon classic formulas due to Newton and a specific solution to an integral equation found by a convergent Neumann series. In $\mathbf{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ we consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq} = a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x, y, z) \to \infty$:

$$\lim a_\nu(p) = 0\colon 0 \leq p \leq 3 \quad : \ \lim a_{pq} = \text{Kronecker's delta function}$$

Thus, L approaches the Laplace operator as (x, y, z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. First we consider the adjoint operator:

$$(0.1) \qquad L^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum\sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$(0.2) \qquad \int L(\phi) \cdot \psi \, dx = \int \phi \cdot L^*(\psi) \, dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x, \partial_x)$ if

$$(0.3) \qquad \psi(0) = \int \Phi \cdot L^*(\psi) \, dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L - \kappa^2$ and a function $x \mapsto \Phi(x; \kappa)$ is a fundamental solution to $L - \kappa^2$ if

$$(0.4) \qquad \psi(0) = \int \Phi(x : \kappa) \cdot (L^* - \kappa^2)(\psi(x)) \, dx$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

$$(*) \qquad \psi(\xi) = \int \Phi(x, \xi; \kappa) \cdot (L^*(x, \partial_x) - \kappa^2)(\psi(x)) \, dx$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$.

1. The construction of $\Phi(x, \xi; \kappa)$.

Consider first the case with constant coefficients where the construction of fundamental solutions already appears in Newton's text-books from 1666. We have the positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq} a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Next, define the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

With these notations Newton's fundamental solution taken at $x = 0$ becomes

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ indeed yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)\}$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and we also notice that $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to find a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we construct below.

1.3 The function $F(x,\xi;\kappa)$**.** For every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3}(a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3}(a_p(x) - a_p(\xi))\frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

With ξ fixed we apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give the existence of positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \ [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy + F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(k-1)}(y,\xi;\kappa)\, dy \quad : \quad k \geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

(i) $$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi\to u$ becomes

(ii) $$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

(iii) $$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi\frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr=$$

(iv) $$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)|\leq\frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

(*) $$|F^{(n)}(x,\xi;\kappa)|\leq\frac{C_1}{|x-\xi|^2}\cdot[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}]^{n-1}\quad\text{hold for every}\quad n\geq 2$$

1.6 Conclusion. With κ_0^* so large that $2\pi C_1^2\cdot C_1^*<\kappa_0^*$ it follows from (*) that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa\geq\kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. Above we have found Ψ which satisfies the integral equation in § 1.5.1 Use Green's formula to show that the function $\Phi(x,\xi;\kappa)$ defined in (1.2.1) gives a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 A final estimate. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa)\quad\text{and}\quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

(1.8.1) $$G(x,\xi;\kappa)=\Phi(x,\xi;\kappa)-H(x-\xi,\xi;\kappa)$$

1.8.2 Exercise. Use the previous constructions to show that for every $0<\gamma\leq 2$ there is a constant C_γ such that

$$\big|G(x,\xi;\kappa)\big|\leq\frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa\geq\kappa_0$. Together with the the inequality for the H-function in (1.4.1) this gives an estimate for the fundamental solution Φ. More precisely we have proved:

1.9 Theorem. *With κ_0^* as above there exist positive constants C and k and for each $0 < \gamma \leq 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} + \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every $\kappa \geq \kappa_0^*$.