Exercises on pA flows

(1) Prove that a pseudo-Anosov flow on a compact 3-manifold is an expansive flow. (Remark: the converse direction is true but hard and follows from work of Paternain and Inaba-Matsumoto)

(2) Fix a topological surface Σ of genus at least 2, and let $M = T^1(\Sigma)$ be the unit tangent bundle. By equipping Σ with various metrics of negative curvature, we produce different examples of geodesic flows on the same (topologically speaking) manifold M. Are these all orbit equivalent to each other? Explain.

(3) Verify that the suspension flow of a hyperbolic linear map of the 2-torus is Anosov, using the following outline. It will also show you that it is an *algebraic example* a flow on $\Gamma\backslash G$ where G is a lie group, Γ a lattice, and the flow ϕ^t is given by right-multiplication by a 1-parameter family in G. (also, please correct any typos in my outline!)

(a) Let A be a hyperbolic linear map. Let $\{x_1, x_2\}$ denote the basis for $\mathbb{R}^2$ consisting of unit eigenvectors of A, with $A(x_1) = \lambda x_1$. The standard Euclidean metric $dx_1^2+dx_2^2$ descends to the torus quotient $\mathbb{T}^2$ so defines a (Euclidean) Riemannian metric there. Now on $\mathbb{T}^2 \times \mathbb{R}$ we fix the metric

$$ds^2 = \lambda^{2t}dx_1^2 + \lambda^{-2t}dx_2^2 + dt^2.$$

Show the quotient map $(x,t) \sim (A(x), t-1)$ is an isometry with respect to this metric, so the metric descends to the quotient.

(b) Using this metric, show the suspension flow is Anosov and describe the stable/unstable distributions

(c) Show that in fact all transformations of the form

$$(x_1, x_2, t) \mapsto (\lambda^{-s}x_1 + a, \lambda^s x_2 + b, s + t)$$

for $(a, b, s) \in \mathbb{R}^3$ are isometries of the metric described above.

The equation

$$(x_1, x_2, t) \mapsto (\lambda^{-s}x_1 + a, \lambda^s x_2 + b, s + t)$$

is the formula for multiplication $(a, b, s) \cdot (x_1, x_2, t)$ in the Lie group Sol, so what you have just studied is a left-invariant metric defining a Sol-structure on $\mathbb{T}^2_A$ for which the suspension flow is given by *right* multiplication by the one-parameter subgroup $(0, 0, t)$.

(4) **[We did this in lecture but you can work slowly through the details if you like]
The *orbit space* of a flow ϕ^t on M is defined as follows: Lift the flow to $\widetilde{\phi}^t$ on the universal cover $\widetilde{M}$. Then, define an equivalence relation on $\widetilde{M}$ by $x \sim y$ if $x = \widetilde{\phi}^t(y)$ for some $t \in \mathbb{R}$. The orbit space is the quotient of $\widetilde{M}$ by this relation.
Describe, concretely
1) the orbit space for geodesic flow on the unit tangent bundle of a hyperbolic surface and
2) the orbit space for the suspension flow of a hyperbolic linear automorphism of the torus.

(5) Prove the following
Theorem. Suppose G is a group with an Anosov-like action on the *trivially bi-foliated* plane ($\mathbb{R}^2$ with the horizontal/vertical foliations). Then this action of G is conjugate to an action by affine transformations.

You will need to use the following theorem of Solodov from 1-dimensional dynamics:
If H is a group of orientation-preserving homeomorphisms of $\mathbb{R}$ such that each nontrivial $h \in H$ acts with at most one fixed point, then H is isomorphic to a

subgroup of affine transformations. Furthermore, if the action is minimal, then it is conjugate to an action by affine transformations.

Bonus: can you characterize which groups of affine transformations are Anosov-like?

(6) Challenging exercise: Some definitions of pseudo-Anosov flow in the literature omit the condition that distinct orbits on stable (respectively unstable) leaves separate in the past (resp. future). Construct an example of a flow on a 3-manifold which satisfies all other parts of the definition but fails this one. For the experts: Note that it does not have the kind of nice properties you'd like, for instance a markov partition...