1D dynamics, Lecture 3:
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Polynomial-like maps vs pruned polynomial-like structure for real analytic maps

20 May 2024

Polynomial-like maps

One of the main ingredients in complex dynamics is the following:

Definition

Assume that U, U' are open set in $\mathbb{C}$. Then $f\colon U \to U'$ is a polynomial-like map if f is a complex analytic map which is branched covering map with a finite number of branch points. So

- f maps ∂U onto $\partial U'$ and maps U onto U'.
- for each $y \in U'$ there exists a neighbourhood V' of y so that $f^{-1}(V')$ has finitely many component and on each component f is either a homeomorphism or locally of the form $z \mapsto z^d$.

For each such f one can definite its filled Julia set as

$$K(f) = \{z \in U; f^n(z) \in U \text{ for all } n \geq 0\}.$$

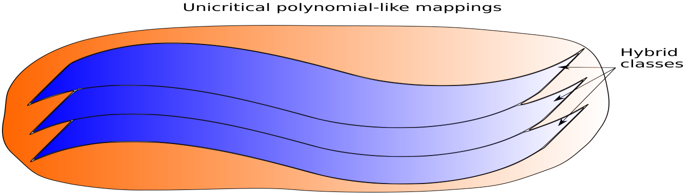
If $K(f)$ may be connected or disconnected. (Draw pictures...)

Manifold structure for quadratic-like mappings

- Let $\mathcal{QL}$ be the space of real quadratic-like mappings, $f : U \to U'$ with $U \Subset U'$.
- Let $\mathcal{C} \subset \mathcal{QL}$ denote the set for which $K(f)$ is connected.
- Hybrid class = Top class + fixing multipliers at periodic attractors.

Theorem (Lyubich)

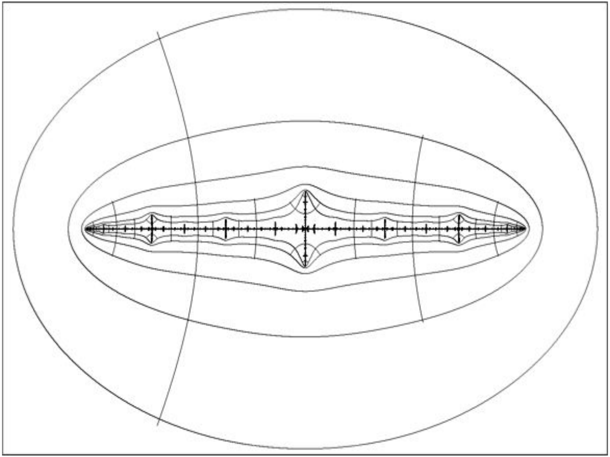
The hybrid class of $f \in \mathcal{C}$ is a connected, codimension-one, complex analytic submanifold of $\mathcal{QL}$.
Moreover, topological conjugacy classes laminate $\mathcal{C}$.



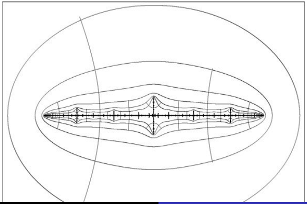
Aim of this lecture

- In the next lecture I will try to explain the proof of the previous theorem, and also try to explain why the proof does not work in our setting.
- Nevertheless we would like to get a structure similar to that of a polynomial-like map.
- For this structure one can obtain the analogue of the previous result (but using a somewhat different approach in the proof).
- That is the purpose of today's talk.

Before discussing the analogue structure, let us explain the following figure, which will be an inspiration for what we will do.

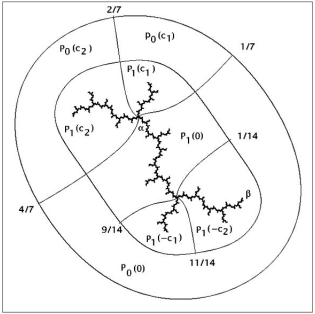


- Here some map quadratic map $f(z) = z^2 + c$ with $c \in \mathbb{R}$ is considered
- its filled Julia set $K(f) = \{z \in \mathbb{C}; \limsup_{n\to\infty} |f^n(z)| \not\to \infty\}$ is drawn.
- Consider the Riemann mapping ψ from $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus K(f)$ so that $\psi(\infty) = \infty$.
- What is drawn in the figure are ψ-images of
    - circles $re^{i\phi}, \phi \in [0, 2\pi]$, $r \geq 1$ fixed (equipotential), and
    - rays $z = re^{i\phi}$, $r \geq 1$ in $\mathbb{C} \setminus \mathbb{D}$ (external ray).
- It turns out that $\psi^{-1} \circ f \circ \psi \colon \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is equal to $z \mapsto z^2$.
- So f maps equipotentials to equipotentials and external rays to external rays.



Puzzle pieces for polynomial maps

- More generally, given a *polynomial* f, there exists a way of constructing *nice sets*, i.e. sets P_n so that no point on the boundary is ever mapped into the interior of P_n.
- This construction uses *external rays and equipotentials* landing on periodic orbits, see Misha's lectures and blackboard. These curves come from the **Böttcher coordinates** near ∞. The partition elements are called *Yoccoz puzzle pieces*.



Aim of this lecture: do an analogous construction for real analytic maps (which are not globally defined).
Issues to overcome:

- how to associate a filled Julia to a real analytic interval map?
- how to obtain a picture as before.

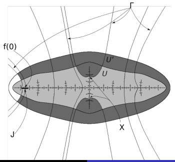
Pruned polynomial-like maps

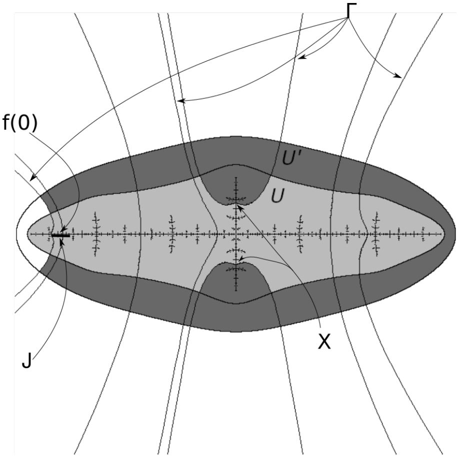
Real analytic maps have a **pruned-polynomial-like** extension:

Theorem (Trevor Clark & SvS)

Associated to $f\colon I \to I$ with only repelling periodic points, $\exists$ open neighbourhoods U, U' of I in the complex plane and a finite union of curves Γ so that f has an extension $F\colon U \to U'$ with

- $U \supset I$, $F(U) = U'$ *and* $F(\partial U) \subset \partial U' \cup \Gamma$;
- *each component of* Γ *is a piecewise smooth arc in* U' *connecting boundary points of* U';
- $F(\Gamma \cap U) \supset \Gamma$
- *each component of* $U' \setminus (\Gamma \cup \mathbb{R})$ *is a quasidisc.*




Γ
f(0)
U'
U
X
J

- This theorem was proved in T. Clark and SvS, *Conjugacy classes of real analytic one-dimensional maps are analytic connected manifolds*, arXiv:2304.00883.
- In A. Avila, M. Lyubich and W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps*, Invent. Math. **154** (2003), 451-550 also gives a a complex extension for real analytic maps. However,
  - their construction requires that there is **only** one critical point, and that this critical point is **quadratic**.
  - the domain of their construction consists of a countable union of open domains, which together do not form a full neigbourhood of I.

  Therefore their extension much less useful and much harder to work with.

What is the aim of this pruned polynomial-like structure?

- Even when f is entire (i.e. defined and holomorphic on the complex plane) we want to 'cut' or 'prune' all dynamics away from the domain that is unrelated to that on I.
- If f is merely real analytic, then the domain of the map is only a small neighbourhood of I. In particular, if the map f may not be holomorphic on a neighbourhood of $f^{-1}(I)$.

What is the benefit of this structure?

- Most techniques that work for quadratic-like maps can also be used in the setting of pruned polynomial-like maps.
- The domain of this pruned polynomial-like extension contains a full neighbourhood of I.

How to obtain this structure?

- Define a geometric object $I \subset K_X \subset \mathbb{C}$ which is full.
- Using the dynamics $f \colon \mathbb{C} \setminus f^{-1}(K_X) \to \mathbb{C} \setminus K_X$ define an external map $g \colon \partial\mathbb{D} \to \partial\mathbb{D}$ (which will have discontinuities);
- Use this to define the pruned polynomial-like extension.

How to obtain pruned polynomial-like maps

We want to construct a Markov structure in a neighbourhood of I. To obtain this, we will consider the Julia set:

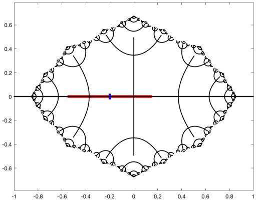
- Assume that we are in the amazing situation that $f\colon I \to I$ extends to a polynomial map $F\colon \mathbb{C} \to \mathbb{C}$
- Also assume that $F^n(z) \to \infty$ when $z \in \mathbb{R} \setminus I$ and all periodic orbits are repelling.
- Then the Julia set is equal to

$$J(f) = \overline{\cup F^{-n}(I)}.$$

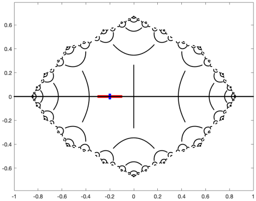
- What to do when $f\colon I \to I$ is real analytic? Prune!

Example:

- let $f(z) = (-c+1)z^2 + c$, normalised so that $F_c(\pm 1) = 1$ with $c = -0.2$.
- take two different intervals J containing c.
- consider the set $K_X(f)$ where $X = \partial J$ (defined on the next page).



(a)



(b)

Pruned Julia Set.

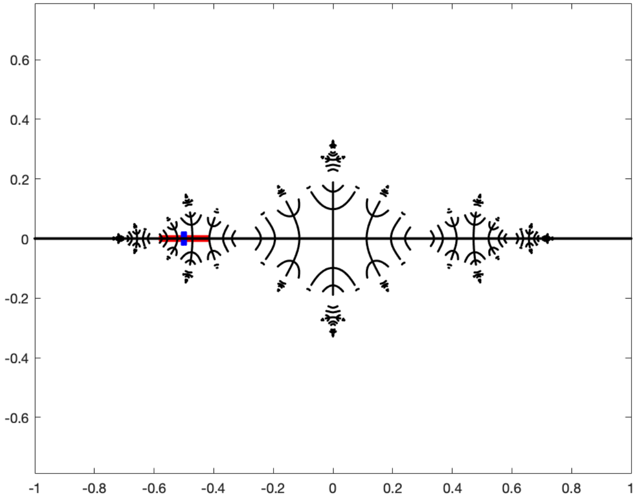
Take

- (real) disjoint interval neighbourhoods $J_1, \dots, J_{\nu'} \subset I$ of the critical values $f(c_1), \dots, f(c_\nu) \in I \subset \mathbb{R}$,
- let $J := \cup J_i$ and
- Let J^{-1} be the union of the connected components of $\overline{f^{-1}(J) \setminus \mathbb{R}}$ containing a critical point.

Now consider the connected component K_n of

$$\cup_{0 \le i \le n} f^{-i} J^{-1} \bigcup I$$

containing I and let $K_X = \overline{\cup K_n}$.


0.6
0.4
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0
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-0.6
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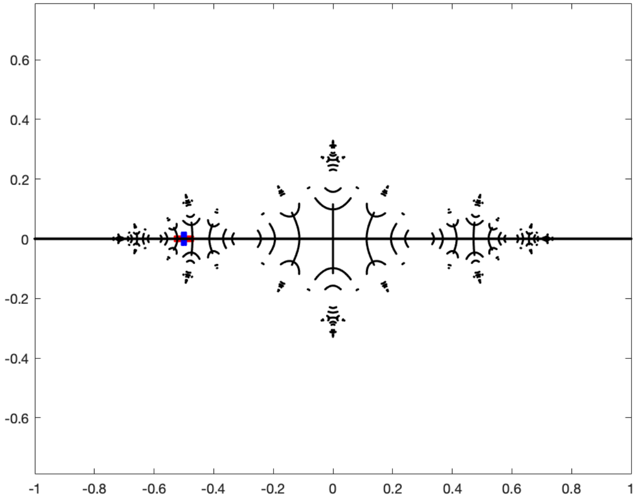
1


0.6
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-0.6
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Pruned Julia Set.

So the Julia set is 'cut' in preimages of the points

$$X = (\partial f^{-1}(J) \setminus \mathbb{R}) \cup \{\text{periodic critical points}\}.$$

Next define the pruned Julia set:

$$K_X(f) = \text{ closure of } \bigcup_{n=0}^{\infty} K_n. \qquad (1)$$

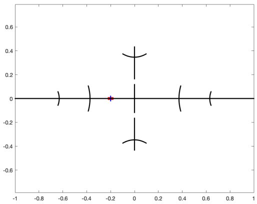
Theorem

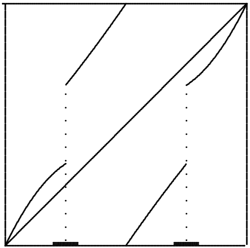
Assuming the intervals J are small enough,

- *the resulting set K_X has no interior, is full and locally connected;*
- $f(K_X) \subset K_X$ *and* $K_X \subset \Omega_a$;
- $f : I \to I$ *has only repelling periodic points* $\implies$ *all periodic points on K_X are repelling;*
- $f : I \to I$ *has only hyperbolic periodic points* $\implies$ *all periodic points on K_X are hyperbolic, and the attracting ones are in I.*

Step 1 in proof: External mapping

- Let $\psi\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus K_X(f)$ be the Riemann mapping and let $\phi = \psi^{-1}$ (is multivalued on $\partial\mathbb{D}$).
- $g = \psi^{-1}\circ f\circ\psi : \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\overline{\mathbb{D}}$ is well-defined near $\partial\mathbb{D}$
- it extends to $\partial\mathbb{D}$ as an analytic map outside $\phi(X)$ and at each of these points g has a discontinuity.
- choose intervals $Y \subset J^{-1}\setminus\mathbb{R}$ containing X.
- in the quadratic case, $\phi(c)$ consists of four points in $\partial\mathbb{D}\setminus\phi(X)$, and their forward orbits don't enter $\hat{Y} := \phi(Y)$.
- Choose Y so that $\partial\hat{Y}$ are pre-periodic points of g.





Intermezzo

There exists a theorem by Mañé which gives expansion:

Theorem

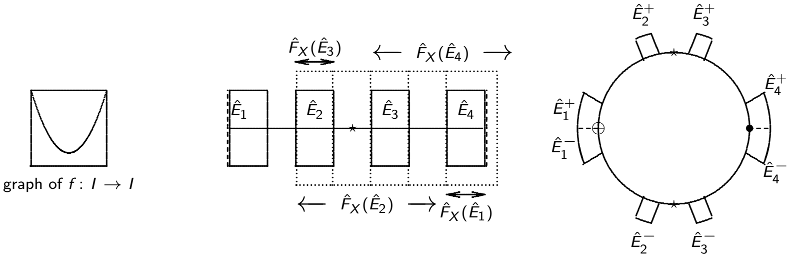
Let f be a C^2 map on the circle (or an interval) without parabolic periodic points. Then for each neighbourhood U of the set of critical points of f there exists $C > 0$ and $\lambda > 1$ so that if

$$z \in \{x; f^i(x) \notin U \text{ for all } 0 \leq i \leq n-1\}$$

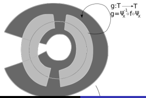
then

$$|Df^n(z)| \geq C\lambda^n.$$

- By the previous theorem $\exists$ an adapted metric on $\partial\mathbb{D} \setminus \hat{Y}$ so that g becomes an expanding map of the circle (if f has no periodic attractors).
- Choose forward invariant rays through each point of $\partial \hat{Y}$.

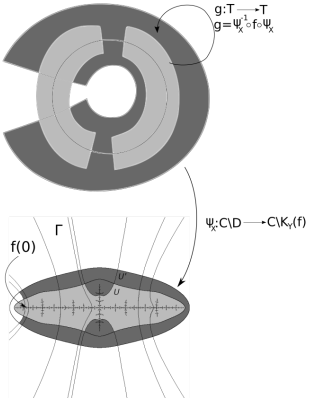


On the next picture we assume that $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$, $\hat{E}_4$ touch.



External mapping: a Markov structure.

- By the previous theorem $\exists$ an adapted metric on $\partial\mathbb{D} \setminus \hat{Y}$ so that g becomes an expanding map of the circle (if f has no periodic attractors).
- Choose forward invariant rays through each point of $\partial\hat{Y}$.
- Using this we obtain sets U and U' as shown:



Pruned polynomial-like mappings

Theorem (Trevor Clark & SvS)

Associated to $f : I \to I$ with only repelling periodic points, $\exists$ neighbourhoods U, U' of I in the complex plane and a finite union of curves Γ so that

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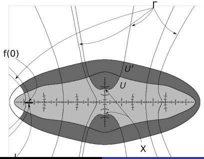
Where you prune, can be encoded in a finite set $Q(F) \subset \partial\mathbb{D}$.



- There exists a similar result if there exists periodic attractors or parabolic periodic points.
- Giving this structure we can start using complex dynamics in the next lecture.
- In the next lecture will give a sketch of the following theorem, but we will assume in this sketch a deep result which will discussed in the final lecture.

Assume that all periodic points of f are hyperbolic.

- $\zeta(f) =$ maximal number of critical points *in the basins* of periodic attractors of f with pairwise disjoint infinite orbits.

Theorem B (Trevor Clark & SvS)

1. *$\mathcal{T}_f^\nu$ is a real analytic manifold.*
2. *$\mathcal{T}_f^\nu \cap \mathcal{A}_a^\nu$ is a real analytic Banach manifold.*
3. *The codimension of $\mathcal{T}_f^\nu$ in the space of all real analytic functions is equal to $\nu - \zeta(f)$.*

Moreover, $\mathcal{T}_f^\nu$ is path connected.

If there are periodic attractors without critical points in its basin we have to adjust this dimension.