1D dynamics, Lecture 2:

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Any real analytic family of interval maps without density of hyperbolicity $\implies$ locally trivial

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- Let I be a compact interval; to be definite, let us take $I = [-1, 1]$.
- Let $\nu \in \mathbb{N}$, and fix a vector $\underline{\nu} = (\ell_1, \ell_2, \dots, \ell_\nu) \in \mathbb{N}^\nu$ with each $\ell_i \geq 2$.
- Let $\mathcal{A}^{\underline{\nu}}$ denote the space of real analytic mappings $f : I \to I$ with $f(\partial I) \subset \partial I$, with precisely ν critical points $-1 < c_1 < c_2 < \cdots < c_\nu < 1$ such that for each $1 \leq i \leq \nu$,

$$Df(c_i) = \cdots = Df^{\ell_i - 1}(c_i) = 0 \text{ and } Df^{\ell_i}(c_i) \neq 0.$$

- For simplicity assume $Df(\pm 1) \neq 0$.
- Throughout we will assume that

$$(f_\lambda)_{\lambda \in N} \text{ is so that } f_\lambda \in \mathcal{A}^{\underline{\nu}},\ (x, \lambda) \mapsto f_\lambda(x) \text{ is real analytic and } N \subset \mathbb{R}^s \text{ is open, connected and } s \geq 1.$$

- For such families critical points depend analytically on λ as the order of critical points is constant and so we can write $c_{i,\lambda}$.

Statement of Theorem A

- A critical point is *hyperbolic* if it is in the basin of a hyperbolic attractor. $f \in \mathcal{A}^{\underline{\nu}}$ is *hyperbolic* if each periodic and each critical point is hyperbolic.
- $(f_\lambda)_{\lambda \in N}$ is *trivial* if at least one of the following holds:
  a) $\exists$ a non-hyperbolic critical point $c_{i,\lambda}$ whose itinerary is the same for all $\lambda \in N$;
  b) there exists $n \geq 1$ and an open set $N' \subset N$ so that f_λ^n has a parabolic fixed point for all $\lambda \in N'$. (VERY SPECIAL)

Here the **itinerary** of a point x w.r.t. to a map f is defined as follows. Associate to c the infinite symbolic sequence

$$i_f(x) := (a_n(x))_{n \geq 0} \in \{c_0, I_0, c_1, I_1, \ldots, I_\nu, c_{\nu+1}\}^{\mathbb{N}}$$

where

$$a_n(x) = \begin{cases} I_i & \text{if} \quad f^n(x) \in (c_i, c_{i+1}), \\ c_i & \text{if} \quad f^n(x) = c_i. \end{cases}$$

Let's discuss part b) of the trivial family property:

b) there exists $n \geq 1$ and an open set $N' \subset N$ so that f_λ^n has a parabolic fixed point for all $\lambda \in N'$. (VERY SPECIAL)

Example of family with locally persistent parabolic periodic point:

$$f_\lambda(x) = x + (x^2 - \lambda)^2.$$

Then $Df_\lambda(x) = 1 + 2x(x^2 - \lambda)$. So when $x_\lambda = \pm\sqrt{\lambda}$ we have

$$f_\lambda(x_\lambda) = x_\lambda \text{ and } Df_\lambda(x_\lambda) = 1 \text{ for } \lambda \geq 0$$

Lemma

If for each $\lambda \in J$, $f_\lambda^n(x) = x$ and $Df_\lambda^n(x) = 1$ implies $D^2 f_\lambda^n(x) \neq 0$ or $D^3 f_\lambda^n(x) \neq 0$ then each locally persistent parabolic point is globally persistent.

Corollary

If $Sf_\lambda < 0$ or f_λ has only real critical points, then in b) one can take $N' = N$.

Let $J \subset \mathbb{R}$.

Theorem A

$$\{\lambda \in J; f_\lambda \textit{ is hyperbolic }\} \quad \textit{is not dense in } J$$

$\Longrightarrow$ *then* $\exists$ *an open interval* $J' \subset J$ *so that* $(f_\lambda)_{\lambda \in J'}$ *is trivial.*

- Theorem A is trivial when f_λ is unicritical, because in that case the only way the itinerary of the critical point can change is by becoming periodic and then the map becomes hyperbolic (unless there are persistent parabolic periodic points).
- If there are two critical points, in principle, it could be that either c_1 or c_2 are hyperbolic, but never simultaneously.
- The proof of Theorem A uses a more sophisticated one-dimensional, using the property that the family is real analytic.

Ingredients in the proof of Theorem A

Definition (Renormalisation interval)

- A closed interval K is called a *renormalisation interval* for f if there exist $s \geq 1$ so that $f^s(K) = K$ and so that K contains a critical point. We call s its *period* if $s \geq 1$ is minimal so that $f^s(K) = K$.
- K is called a *minimal renormalisation interval* if there exists no renormalisation interval $K' \subsetneq K$ for f of the same period as K.

- There are several definitions of renormalisation intervals. We chose this one, because it works better with a later lemma.
- A renormalisation interval K_{λ_0} for f_{λ_0} is *stable* if for λ sufficiently close to λ_0 there exists a renormalisation interval K_λ for f_λ so that K_λ has the same period and depends continuously on λ.

Parameters with good properties

Recall that a residual set is (by definition) the countable intersection of open and dense sets. By Baire Theorem such sets in $\mathbb{R}$ are dense.

The following lemma follows immediately from the definition:

Lemma

One of the following holds:

1. *$\exists$ n and an interval $J' \subset J$ so that f_λ has a parabolic periodic point of period n for each $\lambda \in J'$, or*
2. *there exists a residual set $D \subset J$ so that for each $\lambda \in D$ each periodic point of f_λ is hyperbolic.*

The following two lemmas follow from real analyticity of the family:

Lemma

At least one of the following holds:

- *there exists either an integer* $n > m$ *and a critical point* c_i *so that* $f_\lambda^n(c_i) = f_\lambda^m(c_i)$ *for all* $\lambda \in D' := J$ *or*
- *there exists a residual set* $D' \subset J$ *so that each renormalisation interval for* $\lambda \in D'$ *is stable.*

Lemma

There exists either

- *an integer* $n > 0$ *and critical points* c_i, c_j *so that* $f_\lambda^n(c_i) = c_j$ *for all* $\lambda \in D'' := J$ *or*
- *there exists a residual set* $D'' \subset J$ *so that* $f_\lambda^n(c_i) \neq c_j$ *for all* $1 \leq i, j \leq \nu$, $n > 0$ *and* $\lambda \in D''$.

In particular,

- if $\lambda_0 \in D''$ then for each $N > 0$ there exists $\epsilon > 0$ so that

$$f_\lambda^n(c_i) \neq c_j \text{ for all } |\lambda - \lambda_0| < \epsilon \ , \ 0 < n < N.$$

- So by the implicit function theorem, preimages of $f_\lambda^{-n}(c_j)$ move continuously with $|\lambda - \lambda_0| < \epsilon$.

Initial step in the proof of Theorem A

- Let $0 \leq k(f_\lambda) \leq \nu$ be the number of critical points of f_λ which are contained in the basin of a *hyperbolic* periodic attractors.
- Choose $\lambda_0 \in J$ so that $J \ni \lambda \mapsto k(f_\lambda)$ has a local maximum for $\lambda = \lambda_0$.
- Hence λ_0 has an open and connected neighbourhood $J' \subset J$ so that for all $\lambda \in J'$ all hyperbolic periodic attractors of f_{λ_0} persist and remain hyperbolic, and so that each critical point of f_λ which is contained in the basin of hyperbolic periodic attractor for $\lambda = \lambda_0$ remains in the basin of a hyperbolic periodic attractor for all $\lambda \in J'$.

- Let us call a critical point *hyperbolic* if it is in the basin of a hyperbolic periodic attractor.
- From the above, $c_{i,\lambda}$ is either hyperbolic for all $\lambda \in J'$ or non-hyperbolic for all $\lambda \in J'$.
- Since we assume that f_λ is a non-trivial family, it follows from Lemmas 4, 5 and 6 that we can pick $\lambda_1 \in J \cap D \cap D' \cap D''$ (arbitrarily close to λ_0). Note that $k(\lambda)$ is constant for $\lambda \in J'$ and so we may assume that $\lambda_0 = \lambda_1$.
- Thus from now on we assume that f_{λ_0} has no parabolic periodic points, no non-persistent critical connections and that each renormalisation interval for this map is stable.

Definition

Take a critical point c which is contained in a renormalisation interval K. Then consider a sequence of minimal renormalisation intervals $K \supset K_1(c) \supsetneq K_2(c) \supsetneq \dots$ containing c. This sequence may terminate or be infinite. If it is infinite, then we say that c is *infinitely renormalisable*. By real bounds, see van Strien & Vargas, if c is infinitely renormalisable then $|K_n(c)| \to 0$ as $n \to \infty$.

Pack of periodic points

Definition

Given a repelling periodic point p of period s, let $W^u_\pm(p) = \cup_{n\geq 0} f^{ns}(J)$ where J is a right respectively left one-sided neighbourhood containing in its boundary p so that $f^s|J$ is a diffeomorphism. Let $W^u(p) = W^u_-(p) \cup W^u_+(p)$.

Definition (Pack of periodic points)

- A *pack of periodic points* is a collection of periodic points whose convex hull is a closed (possibly degenerate) interval K for which there exists an integer $s > 0$ so that $f^s(K) = K$, $f^s|K$ is a homeomorphism and so that there exists no larger interval with these properties.
- If x_0 is a periodic point of period s, then denote by $[x_0]$ be the pack of periodic points containing x_0. Define $W^u[x_0] = W^u_+(b) \cup W^u_-(a) \cup O(K)$ where $K = [a, b]$ and $O(K)$ is the forward orbit of K.

Key Lemma

Lemma (Key Lemma)

Let K be a renormalisation interval with $f^s(K) = K$ and assume K contains a critical point which is not mapped into a smaller renormalisation interval $K' \subsetneq K$. Then there exists a repelling periodic point x_0 in the interior of K which is not contained in the boundary of a component of B so that:

1. $W^u[x_0] = K$.
2. *Let $z \in K$ so that $z \notin B$ and so that z never enters any renormalisation interval or pack of periodic points interval $\hat{K} \subsetneq K$. Then the backward orbit of x_0 accumulates onto z.*

Proof: absence of wandering intervals, see

- M. Martens, W. de Melo and S. van Strien. *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. 168 (1992), no. 3–4, 273–318.
- S. van Strien and E. Vargas, *Real bounds, ergodicity, and negative Schwarzian for multimodal maps,* Jour. Amer. Math. Soc. **17**(4) (2004), 749–782.

The proof of Theorem A in the Main Case:

Case (I) There exists a renormalisation interval K whose forward orbit contains at least one non-hyperbolic critical point c_i which never enters a smaller renormalisation interval $K' \subsetneq K$.
In this case we will either

1. find parameters $\lambda \in J'$ arbitrarily close to λ_0 so that $c_i(\lambda)$ is periodic, which gives a contradiction with the choice of λ_0 and the assumption that c_i is non-hyperbolic, or
2. find an interval $J'' \subset J'$ so that the itinerary of $c_i(\lambda)$ and also the itinerary of $c_j(\lambda)$ for each j with $c_j \in \omega(c_i)$ remains constant for all $\lambda \in J''$ implying Theorem A.

- Write $K_{\lambda_0} = K$, let s be so that $f^s_{\lambda_0}(K_{\lambda_0}) = K_{\lambda_0}$ and let $x(\lambda_0)$ be the fixed point of $f^s_{\lambda_0} \colon K_{\lambda_0} \to K_{\lambda_0}$ from the Key Lemma.
- Hence, for each neighbourhood of U of $x(\lambda_0)$ there exists $q > 0$ so that $f^q_{\lambda_0}(U) = K$. In particular there exists a fundamental domain $D_{\pm,\lambda_0}$ for $x(\lambda_0)$ and $q > 0$ so that $f^q_{\lambda_0}(D_{\pm,\lambda_0}) \ni c_i$ for each critical point c_i in $O(K)$.
- Since K_{λ_0} is stable we may shrink the interval $J' \ni \lambda_0$ so that there exists a renormalisation interval K_λ containing $c_i(\lambda)$ for $\lambda \in J'$ which is a continuation of the renormalisation interval K_{λ_0} for f_{λ_0}.
- Let $x(\lambda)$ and $D_{\pm,\lambda}$ be the periodic point and fundamental domains of f_λ for λ near λ_0 depending real analytically on λ for $\lambda \in J'$ where $J' \subset J$ is a sufficiently small neighbourhood of λ_0. We still have that

$$f^{q_\pm}_\lambda(D_{\pm,\lambda}) \ni c_i(\lambda) \tag{1}$$

for each $\lambda \in J'$ provided we take the parameter interval J' sufficiently small.

Case Ia: There exists a parameter interval $J'' \subset J'$ containing λ_0 so that for any $\lambda \in J''$ no iterate of any non-hyperbolic critical point $c_i(\lambda)$ in K_λ is equal to $x(\lambda)$.
Under this assumption, the itinerary of $c_i(\lambda)$ in K_λ is constant for $\lambda \in J''$. Note that this implies the conclusion of Theorem A.

Case Ib: There exists a parameter interval $J'' \subset J'$ and an integer r so that $f_\lambda^r(c_i(\lambda)) = x(\lambda)$ for all $\lambda \in J''$. Then again the itinerary of $c_i(\lambda)$ is constant for $\lambda \in J''$, and the conclusion of Theorem A holds again.

Case Ic: Ia and Ib are not satisfied.

- Then there exists a sequences $\lambda_n \to \lambda_0$, $t_n > 0$ with $t_n \to 0$ and integers l_n so that

$$f_{\lambda_n}^{l_n}(c_i(\lambda_n)) = x_0(\lambda_n)$$

$$f_{\lambda}^{l_n}(c_i(\lambda)) \neq x_0(\lambda) \text{ for each } \lambda \in (\lambda_n, \lambda_n + t_n).$$

  For $\lambda \in (\lambda_n, \lambda_n + t_n)$ close enough to λ_n there exists $k_{n,\lambda}$ and $0 < t_n' < t_n'' < t_n$ so that

$$f_{\lambda}^{l_n + k_{n,t}s}(c_i(\lambda)) \in \cup_{\pm} D_{\pm,\lambda} \text{ for } \lambda \in (\lambda_n + t_n', \lambda_n + t_n'').$$

  and $f_{\lambda}^{l_n + k_{n,t}s}(c_i(\lambda))$ is equal to the two endpoints of $D_{\pm,\lambda}$ for $\lambda \in \{\lambda_n + t_n', \lambda_n + t_n''\}$.

- On the other hand, we had

$$f_{\lambda}^{q_{\pm}}(D_{\pm,\lambda}) \ni c_i(\lambda) \qquad (2)$$

- Let us write this as $f_\lambda^{-q_\pm}(c_i(\lambda)) \in D_{\pm,\lambda}$ for all $\lambda \in J'$, where we observe that this point moves real analytically with $\lambda \in J'$, provided we choose J' sufficiently small. This holds as no iterate of c_i hits another critical point c_j under the map f_{λ_0} as $\lambda_0 \in D''$, see Lemma 6.
- It follows by continuity that there exists $\lambda \in (\lambda_n, \lambda_n + t_n)$ so that

$$f_\lambda^{r_n+k_{n,\lambda}s}(c_i(\lambda)) = f_\lambda^{-q_\pm}(c_i(\lambda))$$

i.e.,

$$f_\lambda^{r_n+k_{n,\lambda}s+q_\pm}(c_i(\lambda)) = c_i(\lambda).$$

But this contradicts that c_i is non-hyperbolic and the choice of λ_0.

Theorem A''

Given a family f_λ, $\lambda \in J \subset \mathbb{R}$, for which we do not have density of hyperbolicity, there exist and open interval $J' \subset J$ and closed disjoint intervals $I_{1,\lambda}, \ldots, I_{r,\lambda}$, depending analytically on $\lambda \in J'$ so that

1. *all the maps*

$$f_\lambda \colon I_{1,\lambda} \cup \cdots \cup I_{r,\lambda} \to \mathbb{R}$$

   with $\lambda \in J'$ are topologically conjugate to each other;
2. *each critical point in $\cup_i I_{i,\lambda}$ is non-hyperbolic and their forward orbit under f_λ remains in $\cup_i I_{i,\lambda}$ for each $\lambda \in J'$;*
3. *each periodic point in $\cup I_{i,\lambda}$ is hyperbolic for each $\lambda \in J'$;*
4. *for each $\lambda \in J'$, the boundary points of $I_{i,\lambda}$ are all eventually mapped on a repelling periodic point.*

Let us discuss (3) in the previous theorem

The map f_{λ_0} has only hyperbolic periodic points. However, there is no reason for all periodic points to remain hyperbolic for all nearby maps. To conclude this, we will use the property from Theorem A that the itinerary of the non-hyperbolic critical points remains the same within the family f_λ when $\lambda \in J'$ and the following theorem.

Theorem

Let $f \in \mathcal{A}^{\underline{\nu}}$ be map of an interval or circle. Suppose that f does not have neutral periodic orbits. Let c be a non-hyperbolic critical point of f. Then there exist a neighbourhood $\mathcal{U} \subset \mathcal{A}^{\underline{\nu}}$ of f and a neighbourhood U of c such that if $g \in \mathcal{U}$ and $g^n(x) \in U$ for some and $n \geq 0$, then $Sg^{n+1}(x) < 0$.

Corollary

Assume that f has no parabolic periodic points. Then nearby maps with attracting periodic points of high period must have a critical point in their basin.

- M. Martens, W. de Melo and S. van Strien. *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. 168 (1992), no. 3–4, 273–318.
- S. van Strien and E. Vargas, *Real bounds, ergodicity, and negative Schwarzian for multimodal maps,* Jour. Amer. Math. Soc. **17**(4) (2004), 749–782.
- O. Kozlovski, *Periodic attractors of perturbed one-dimensional maps*, Ergodic Theory Dynam. Systems 33 (2013), no. 5, 1519–1541.

The next lecture will discuss the first ingredient of the theorem the space of conjugate maps forms a real analytic manifold.