# 1D dynamics, Lecture 1:
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# density of hyperbolicity revisited

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Purpose of today's talk:

- Survey some the theorems in 1D, including a new theorem about the density of hyperbolicity in one parameter families.

In each of the remaining lectures, I will introduce one new tool and then sketch how this tool is used to prove the above theorems.

# 1D maps with good behaviour

1D maps can have simple and also very complicated dynamics.

**Simple HYPERBOLIC behaviour** for

- Lebesgue almost all points $x$ converge to the orbit of a hyperbolic periodic point $p$, so that $|Df^n(p)| \neq 1$.

Examples:
- $f(x) = 2x(1-x)$ (attracting fixed point) or
- $f(x) = ax(1-x)$ with $a = 3.8284 + \epsilon$ (period three attractor).

**Statistically simple behaviour**

- Lebesgue almost all points $x$ converge to a union of intervals $K$ and there exists an absolutely continuous invariant measure on $K$ so that for a.e. x, $\sum_{i=0}^{n} \delta_{f^i(x)} \to \mu$ in the weak topology.

Example:
- $f(x) = 4x(1-x)$

# 1D maps with bad behaviour

**Example 1**
– [1] $\exists f$ with a $f$-forward invariant Cantor set $\Lambda$ which is a **wild attractor**. That is $\{x; f^n(x) \to \Lambda\}$

- has full Lebesgue measure;
- but is topologically meagre.

**Example 2**
– [2] $\exists f$ which has **strange resp. no physical measures**.

- $(1/n)\sum_{i=0}^{n} \delta_{f^i(x)}$ converges to $\delta_p$ for a.e. $x$, where $p$ is a repelling periodic point.
- or, alternatively $\exists f$, s.t. for a.e. $x$, $\sum_{i=0}^{n} \delta_{f^i(x)}$ does **not** have a limit.

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[1] Bruin, Keller, Nowicki, SvS, Annals 1996

[2] Hofbauer, Keller 1990

Main Purpose of this talk: to discuss theorems which state that **most maps are hyperbolic**:

**Definition** $f$ is **hyperbolic** $\iff$ non-wandering set

$\Omega = \Omega_1 \cup \Omega_2$: $\Omega_1$ exp. expanding, $\Omega_2$ exp contracting (finite)

$\iff$ a.e. $x$ tends to some (hyperbolic) attracting periodic point $\iff$

all critical points are in basins of hyperbolic periodic attractors **and**

all periodic orbits are hyperbolic.

Here a critical point is a point $x$ so that $Df(x) = 0$.

- For people in complex dynamics or if $Sf < 0$, they usually omit in the last sentence the part that all periodic points are hyperbolic because this follows from the first part.
- General interval maps, may have periodic attractors that do not have critical points in their basin.

# Open problems in dynamics

The following questions have been open for many years:

- **Closing Lemma:** Let $x$ be a recurrent point of $f$. Is it possible to approximate $f$ in the $C^2$ topology by a diffeomorphism for which $x$ is periodic?
  In **real dimension 1** the answer is yes: even in the $C^\infty$ topology.
- **Density of hyperbolicity:** Let $M$ be a manifold with $\dim(M) \geq 2$ and let $f_0$ be a $C^k$ diffeomorphism on $M$.
  - Can one approximate $f$ by a hyperbolic diffeomorphism?
  Since the late 1960's it is known that the answer is: NO.
  In **real dimension 1** the answer is yes: even in the $C^\infty$ topology.
- **Palis conjecture:** By a diffeo with finite number of attractors? NOT KNOWN!
  In **real dimension 1** only a partial solution known.

# Density of hyperbolicity is open for complex quadratic maps

Fatou posed the analogous conjecture for maps of the interval or on the Riemann sphere.

**Conjecture (Fatou 1920)**

*Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then there exists a polynomial $g$*

- *of the same degree and*
- *whose coefficients are arbitrarily close to those of $f$,*

*for which $g$ is hyperbolic.*

Still wide open!!!

# Density of hyperbolicity in real dimension 1

- As mentioned, the Fatou conjecture in $\mathbb{C}$ is still wide open.
- For real dimension $= 1$, density of hyperbolicity was part of Smale's list of questions for the 21st century.
- Is was proved about 17-27 years ago:

**Theorem (Kozlovski, Shen and SvS, Annals 2007a)**

*Within the space of $C^k$ interval maps ($k \in \mathbb{N}$ or $k = \omega$), hyperbolic maps are dense and the $C^k$ closing lemma holds.*

Note: this was proved before for $z^2 + c$, $c \in \mathbb{R}$, or $ax(1-x)$, $a \in (0, 4]$, by

- Graczyk & Swiatek, Annals of Math 1997 and, independently, by
- Lyubich, Acta Math 1997.
- In the quadratic case a miracle occurs: 2/2=1.
- Our proof necessarily follows a completely different approach.

- **Density hyperbolicity: local vs global**
- Pugh's $C^1$ closing lemma uses **local perturbations** to make a recurrent point periodic for a $C^1$ nearby diffeo: pick a 'closest' return time and then to do a local perturbation.
- This 'local' approach was also attempted (many times) in the one-dimensional case. In 1971 Jacobson: $C^1$ closing lemma. In 2004, Shen: $C^2$ closing lemma for maps satisfying 'bounded geometry'. Attempts to generalise this failed.
- **Palis conjecture:**
- There are many people who have worked on the **Palis conjecture** in dim $\geq 2$ (in $C^1$ topology): Yoccoz, Palis, Pujals. Samborino, Crovisier, Bonatti, Matheus, Avila, etc...
- **Counter example (to the original Palis conjecture):** Berger.
- dim $= 1$: results by Avila, Lyubich, Shen, Bruin, SvS but **only in the unicritical case**.

# Global approach in the quadratic 1D case

- Use global perturbations and rigidity.
- In the case of real quadratic maps, this relies on a trivial lemma plus a deeper result about qs-rigidity:

**Lemma**

*Consider $f_c(z) = z^2 + c$ with $c$ is real. If $f_c$ is not hyperbolic and cannot be approximated by a hyperbolic quadratic map, then there exists a maximal interval $[c_0, c_1] \ni c$ with $c_0 \neq c_1$ so that all maps $f_t$ with $t \in [c_0, c_1]$ are topologically conjugate.*

**Theorem (quasisymmetric rigidity)**

*$f_{c_0}, f_{c_1}$ are quasisymmetrically conjugate. (Defn: see blackboard)*

**Corollary (Using Measurable Riemann Mapping Theorem)**

*There exists $(c_0', c_1') \supset [c_0, c_1]$ so that $f_t$ is topologically conjugate to $f_t$ for each $t \in (c_0', c_1')$.*

# Global approach in the almost general polynomial 1D case

If **ALL critical points of a polynomial are real**:

**Theorem (quasisymmetric rigidity: Kozlovski, Shen, SvS, Annals 2007a)**

*Assume that $f, g$ are real analytic interval maps which are topologically conjugate (and the order of critical points are the same). Then they are quasisymmetrically conjugate.*

- Main tool: enhanced nest to obtain control of geometry of puzzle pieces.[3]
- Within **space of SUCH polynomials** we have qs rigidity $\implies$ rigidity.
- That is, $f, g$ qs conjugate polynomials $\implies$ $f, g$ affinely conjugate or $\exists$ periodic attractors.

[3] KSvS, Annals 2007a. The enhanced is useful for many applications, also to complex dynamical systems, see Kozlovski, SvS, 2009. For an exposition and some further results, see Clark, Drach, SvS, Arnold J. 2022.

In the setting of polynomials with only real critical points, by induction on the number of critical points **rigidity** $\implies$ **density of hyperbolicity**.

Indeed:

- First induction step: $\nexists$ open set $\mathcal{U} \subset P^d$ without an critical relations or per. attr., because otherwise all $g \in \mathcal{U}$ are topologically conjugate, which contradicts rigidity.
- Next induction step: Now consider a subspace of polynomials with one (or several) critical relation, and repeat the argument.
- In each step it is crucial that one has rigidity, not just qs rigidity.

This techniques breaks down if
- you have a polynomial with non-real critical points,
- you have a transcendental or not globally defined map.

# Approach in the general polynomial 1D case

For a **general real polynomial**, or a **real analytic interval map** one can combine **local perturbations** with **global rigidity**.

**Theorem (Density of hyperbolicity: Kozlovski, Shen, SvS 2007b)**

- *Real hyperbolic polynomials of degree $d$ are dense in the space of all polyonomials of degree $d$;*
- *Hyperbolicity is dense in the space of $C^\infty$ interval maps (in the $C^\infty$-topology).*

One step in the proof goes as follows:

- Fix big ball s.t. $f: B \to f(B)$ is polynomial-like of degree
- approximate $f$ by a suitable smooth hyperbolic map $g$ (this is the main step).
- approximate $g$ by a holomorphic $\tilde{g}$ of degree $d' >> d$;
- $\tilde{g}: B \to \tilde{g}(B)$ is qc conjugate to a nearby polynomial $P$ of degree $d$ (**straightening theorem**);

# Why do we want to do better than this?

The techniques in that paper do *not work* for

- specific families of maps (which you cannot choose);
- space of transcendental maps.

# The setting for a new density theorem

Let $\mathcal{A}^{\underline{\nu}}$ be the space of real analytic maps of $I$ with $\nu$ critical points $c_1 < \cdots < c_\nu$ with degrees $\ell_1, \ldots, \ell_\nu$.

What is the **itinerary** of a point $x$? As before, let $c_0 < c_1 < \cdots < c_\nu < c_{\nu+1}$ be the critical points of $f : I \to I$, where $c_0, c_{\nu+1}$ are the left and right endpoint of $I$.
Associate to $c$ the infinite symbolc sequence

$$a_n \in \{c_0, I_1, c_1, I_2, \ldots, I_\nu, c_{\nu+1}\}^{\mathbb{N}}$$

where

$$a_n = \begin{cases} I_i & \text{if} \quad f^n(x) \in (c_i, c_{i+1}), \\ c_i & \text{if} \quad f^n(x) = c_i. \end{cases}$$

# A new density theorem

**Definition:** a family $(f_\lambda)_{\lambda\in J}$ is called **trivial** if there exists a non-hyperbolic critical point $c$, so that one of the following hold:

(a) $J \ni \lambda \mapsto$ itinerary of $c$ is constant;

(b) $\exists$ $n$ and open $J' \subset J$ s.t. $f_\lambda^n$ has a parabolic fixed point $\forall \lambda \in J'$.

If, for example, $Sf_\lambda < 0$ then

(b') can be replaced by $\exists$ $n$ s.t. $f_\lambda^n$ has a parabolic fixed point $\forall \lambda \in J$.

**Theorem (Density of hyperbolicity within families, vS2024)**

*Let $(f_\lambda)_{\lambda\in J}$ be a real analytic one parameter family of real analytic maps which is non-trivial.*
*Then the set of hyperbolic parameters $H$ is dense in $J$.*

# Another version of the new density of hyperbolicity theorem

**Theorem (2nd version Density of hyperbolicity within families, vS2024)**

*Let $(f_\lambda)_{\lambda \in J}$ be a real analytic one parameter family of real analytic maps.*
*Assume that no critical point has the same itinerary for all $\lambda \in J$.*
*Then the set of semi-hyperbolic parameters $H$ is dense in $J$.*

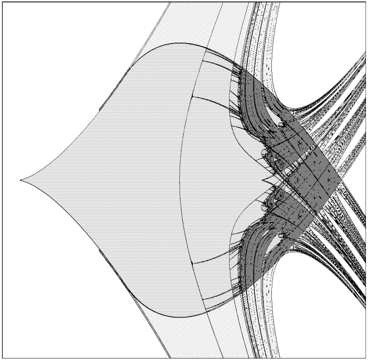
**Definition** $f$ **is called semi-hyperbolic** if all critical points are in the basin of a periodic attractor (possibly parabolic). Here we only consider the real critical points.

Often the assumption of non-triviality is easy to check:

**Application of the Main Theorem**

Take cubic family $f_{A,b}(z) = z^3 - 3Az + b$, $A > 0, b \in \mathbb{R}$,with critical points $\pm\sqrt{A}$.

Figure below (taken from a paper by Milnor) shows bifurcation diagram for $(A, b) \in [-0.4, 1.1] \times [-1, 1]$. **Within any real analytic curve, one has density of hyperbolicity.**



# One page sketch of proof

hyperbolicity is not dense

$\Downarrow$ Theorem A

$\exists J' \subset J$ and a critical point $c$ with constant kneading invariant

i.e. "locally trivial'

$\Downarrow$ Theorem B

globally trivial or parabolic periodic point appears

$\Downarrow$ Theorem D

Statement of New Theorem

# First ingredient in the proof: non-density of hyperbolicity $\implies$ family is locally trivial

**Theorem A (non-density of hyperbolicty $\implies$ locally trivial)**

*Take a one parameter family $f_\lambda$, $\lambda \in J$ of real analytic maps in $\mathcal{A}^{\underline{\nu}}$. Assume that $J$ is an interval of parameters so that*

$$\lambda \in J \implies f_\lambda \text{ is non-hyperbolic}$$

*Then $\exists$ an interval $J' \subset J$ and $i$ so that the the itinerary of the critical point $c_i(\lambda)$ is the same for all $\lambda$.*

# Second ingredient in the proof: family is locally trivial $\implies$ globally trivial

**Case 1: critical point has finite orbit (trivial case)**

**Theorem**

*Suppose that some critical point $c_i$ of $f_\lambda$ is periodic or pre-periodic $\forall \lambda \in J'$, where $J'$ is an open subinterval of $J$. Then it is periodic for* **all** $\lambda \in J$.

Proof is easy: Suppose that for some $q > r \geq 0$ one has $f_\lambda^q(c_i) = f_\lambda^r(c_i)$ for all $\lambda \in J'$. Then this equation holds for all $\lambda$ (by real analyticity).

Note that this argument does **not** use that the periodic point on which $c_i$ lands is hyperbolic. In fact, it shows that the periodic point $f_\lambda^r(c_i)$ must persist for all $\lambda \in J$. (Corollary: no parabolic saddle-node bifurcation can occur.)

# Second ingredient in the proof: family is locally trivial $\implies$ globally trivial

**Case 2: critical point is infinite**

**Theorem B**

*Suppose that there exists $J' \subset J$ so that*

- $\forall \lambda \in J'$ *the itinerary of* $c_i$ *is the same;*
- $\omega(c_i(\lambda))$ *does not contain parabolic periodic points for any* $\lambda \in J$.

*Then the itinerary of* $c_i$ *is the same for* **all** $\lambda \in J$.

Case 2 is quite a bit more subtle that Case 1. Proof uses that certain infinite dimensional manifolds are analytic. To discuss this we need to a digression.

# Tool in Second ingredient: $\mathcal{T}_f^\nu$ is an analytic manifold.

- $\mathcal{T}_f^\nu = \{g \in \mathcal{A}^\nu; g$ topologically conjugate to $f$ **and** all periodic points of $g$ are hyperbolic $\}$.

  - $\zeta(f) =$ maximal number of critical points *in the basins* of periodic attractors of $f$ with pairwise disjoint infinite orbits.

**Theorem B (Trevor Clark & SvS)**

1. *$\mathcal{T}_f^\nu$ is a real analytic manifold.*
2. *$\mathcal{T}_f^\nu \cap \mathcal{A}_a^\nu$ is a real analytic Banach manifold.*
3. *The codimension of $\mathcal{T}_f^\nu$ in the space of all real analytic functions is equal to $\nu - \zeta(f)$.*

*Moreover, $\mathcal{T}_f^\nu$ is path connected.*

If there are periodic attractors without critical points in its basin we have to adjust this dimension.

# A variant of the previous theorem that is needed in the Second ingredient: $\mathcal{T}_f^{\nu}(c_{i_1}, \ldots, c_{i_s})$ is an analytic manifold

In the conclusion of Theorem A we only obtained that the itinerary of only **some** of the critical points are fixed. So we need the following variant:

- Pick $s$ real citical points $c_{i_1}, \ldots, c_{i_s}$, s.t. $c \in \omega(c_{i_j}) \implies c = c_{i_k}$ for some $k$.
- Assume all periodic points in $\omega_f(c_{i_j})$ are hyperbolic.
- Define

$$\mathcal{T}_f^{\nu}(c_{i_1}, \ldots, c_{i_s}) = \{g \in \mathcal{A}^{\nu} \text{ s.t.} i_g(c_{i_j}) = i_f(c_{i_j}), j = 1, \ldots, s \text{ and}$$

$$\text{all periodic points in } \omega_g(c_i) \text{ are hyperbolic } \}.$$

**Theorem C**

*$\mathcal{T}_f^{\nu}(c_{i_1}, \ldots, c_{i_s})$ is a real analytic manifold.*

# Third ingredient in the proof: the real analytic manifold is not real analytic at its boundary

**Theorem D (Avoiding parabolic periodic points)**

*Assume that there is a closed interval $J' \subset J$ and a critical point $c_i$ so that the itinerary of $c_{i,\lambda}$ is the same for all $\lambda \in J'$. Then the following holds: if for some $\lambda_0 \in J'$ there exists a periodic point in $\omega(c_{i,\lambda_0})$ which is parabolic, then this periodic point is parabolic for each $\lambda \in J'$.*

In other words:
The above manifold is not real analytic at its boundary.

# Plan of remaining lectures

In the remaining lectures I will expand on the proof. In the process I will discuss some of the Main Tools in 1D.

1. Survey on various theorems in 1D, including a new theorem about the density of hyperbolicity in one parameter families.
2. Real tools: Absence of wandering intervals and the first step in the proof of the new theorem (Theorem A).
3. Complex tools, part I: Polynomial-like maps, puzzle pieces, pruned polynomial-like structure.
4. Complex tools, part II: Measurable Riemann Mapping Theorem, mating and the conjugacy class of a real analytic map is a real analytic manifold.
5. Complex tools, part III: Quasi-symmetric rigidity and the enhanced nest.

# Literature


- A. Avila, M. Lyubich and W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps*, Invent. Math. **154** (2003), 451-550.

* T. Clark and SvS, *Conjugacy classes of real analytic one-dimensional maps are analytic connected manifolds*, arXiv:2304.00883. Submitted for publication.

- T. Clark, K. Drach, O. Kozlovski and S. van Strien, *The dynamics of complex box mappings*, Arnol'd Math. J. **8** (2022), no. 2, 319–410.
- O. Kozlovski, W. Shen, S. van Strien, *Rigidity for real polynomials*, Ann. of Math. **165** (2007), 749-841.
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- M. Lyubich, *Dynamics of quadratic polynomials I-II,* Acta Math. **178** (1997), 185-297.
- M. Lyubich, *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture,* Ann, of Math. **149** (1999), 319-420.

* SvS, *Density of hyperbolicity holds within families of real analytic one-dimensional maps*. About to be put on ArXiv.

# Python code for period doubling diagram

Exercise: improve the following code (which I found on the internet).

```
import numpy as np
import matplotlib.pyplot as plt
Many =50000
x = np.random.rand(Many)
r = np.linspace(0,4.0, num= Many)

for i in range(1, 54):
   x_a = 1-x
   Data= np.multiply(x,r)
   Data= np.multiply(Data, x_a)
   x = Data

plt.title(r'Logst: $x_{n+1} = a  x_n  (1-x_n).$  n = '+ str(i) )
plt.ylabel('x-Random number')
plt.xlabel('r-Rate')
        plt.scatter(r, Data, s=0.1, c='k')
plt.show()
plt.savefig(str(i) + " Logistic Map.png", dpi = 300)
plt.clf()
```